University of Toronto Mississauga

Mathematical and Computational Sciences
MAT133Y5Y- Term Test 2
Test Booklet
Duration - 110 minutes
No Aids Permitted

Surname/Family Name: ______________________

First/Given Name: ______________________

Student Number: ______________________

UTORid: ______________________

Email: ______________________

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in. You must also have the corresponding Multiple Choice Booklet, which contains the multiple choice questions needed for Question 5 (page 6). No aids are permitted on this examination. Examples of illegal aids include, but are not limited to textbooks, notes, calculators, or any electronic device.

Each part of Question 5 includes five multiple choice answers, of which only a single response is correct. All answers should be indicated on the included scantron sheet (page 6). When filling out the scantron:

- Use a #2 pencil to bubble answers (do not use ink). A #2 pencil ensures that the marks are dark enough.
- Ensure that bubbles are completely filled and make complete erasures.

Any failure to follow these instructions which results in the scanner not being able to read your work will result in a 3 mark deduction. You will not be penalized for incorrect answers.

Your solutions to Problems 1 - 4 must be written in the space provided. No additional paper may be submitted for grading. Do not, under any circumstances, write on the QR code in the top right corner of each page. **Doing so will result in you receiving a zero (0) on this test.**

Unless otherwise indicated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work, will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

The Multiple Choice Booklet contains a **formula sheet** on its last page. Additionally, the Multiple Choice Booklet is printed one-sided, and may thus be used for rough work.

1. You are a bouquet designer for the Grassmann Flower Company. You've come up with a dazzling new arrangement, which you are calling the "What in carnation!" bouquet. To fulfill an order for your new bouquet, you need

2100 orchids, 900 lilies, 2400 carnations, and 1200 hydrangeas.

Your company has four suppliers, each of which ships flowers in bunches:

Alexey's Awesome Amaryllis: 1 bunch = 2 orchids and 1 lilies.

David's Dazzling Daisies: 1 bunch = 1 orchid and 1 carnation.

Srishti's Superb Snapdragons: 1 bunch = 3 carnations and 1 hydrangeas.

Paula's Perfect Peonies: 1 bunch = 3 orchids, 2 lilies, 2 hydrangeas.

Determine, by any means available to you, how many bunches you should buy from each supplier in order to make the "What in carnation!" bouquet.

Solution: Let $A, D, S,$ and P be the number of bunches that we purchase from Alexey, David, Srishti, and Paula respectively. If we let $\begin{bmatrix} O & L & C & H \end{bmatrix}^T$ denote the number of orchids, lillies, carnations, and hydrangeas that come in a bunch, then

$$A\begin{bmatrix}2\\1\\0\\0\end{bmatrix} + D\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + S\begin{bmatrix}0\\0\\3\\1\end{bmatrix} + P\begin{bmatrix}3\\2\\0\\2\end{bmatrix} = \begin{bmatrix}2100\\900\\2400\\1200\end{bmatrix}.$$

This is solved by row reducing the augmented matrix whose coefficients are the columns above, as follows:

$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & 3 & 2100 \\ 1 & 0 & 0 & 2 & 900 \\ 0 & 1 & 3 & 0 & 2400 \\ 0 & 0 & 1 & 2 & 1200 \end{array}\right] \xrightarrow{-2R_2+R_1\to R_1} \left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 300 \\ 1 & 0 & 0 & 2 & 900 \\ 0 & 1 & 3 & 0 & 2400 \\ 0 & 0 & 1 & 2 & 1200 \end{array}\right] \xrightarrow{-R_1+R_3\to R_3} \left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 300 \\ 1 & 0 & 0 & 2 & 900 \\ 0 & 0 & 3 & 1 & 2100 \\ 0 & 0 & 1 & 2 & 1200 \end{array}\right]$$

$$\xrightarrow{-3R_4+R_3\to R_3} \left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 300 \\ 1 & 0 & 0 & 2 & 900 \\ 0 & 0 & 0 & -5 & -1500 \\ 0 & 0 & 1 & 2 & 1200 \end{array}\right] \xrightarrow{-1/5R_3\to R_3} \left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 300 \\ 1 & 0 & 0 & 2 & 900 \\ 0 & 0 & 0 & 1 & 300 \\ 0 & 0 & 1 & 2 & 1200 \end{array}\right]$$

$$\xrightarrow[\substack{R_3+R_1\to R_1\\ -2R_3+R_2\to R_2}]{-2R_3+R_4\to R_4} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 600 \\ 1 & 0 & 0 & 0 & 300 \\ 0 & 0 & 0 & 1 & 300 \\ 0 & 0 & 1 & 0 & 600 \end{array}\right] \xrightarrow[R_3\leftrightarrow R_4]{R_1\leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 300 \\ 0 & 1 & 0 & 0 & 600 \\ 0 & 0 & 1 & 0 & 600 \\ 0 & 0 & 0 & 1 & 300 \end{array}\right]$$

Thus we need to purchase 300 bunches from Alexey and Paula, and 600 bunches from David and Srishti.

Rubric:

- Two (2) marks for setting up the problem
- Six (6) marks for a correct row reduction, with part marks awarded subject to how far the student got before an arithmetic error was made.
- Two (2) marks for the correct answer.

2. Consider the matrix $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 4 & 0 & 5 \end{bmatrix}$.

(i) (4 points) Compute $\det(A)$.

Solution: It's easiest to expand along the third column, which gives

$$\begin{aligned} \det(A) &= (-1)^{1+3}(0)\det\begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix} + (-1)^{2+3}(0)\begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix} + (-1)^{3+3}(5)\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \\ &= 5 \times 2 = 10. \end{aligned}$$

Rubric:

- Two (2) marks for correctly setting up the determinant.
- Two (2) marks for a correct answer.

(ii) (4 points) Compute A^{-1}.

Solution: We set up an augmented matrix $\begin{bmatrix} A \mid I_3 \end{bmatrix}$ and row reduce:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 4 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-4R_1+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & -12 & 5 & -4 & 0 & 1 \end{array}\right] \xrightarrow{5R_2+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & -4 & 6 & 1 \end{array}\right]$$

$$\xrightarrow[1/5R_3\to R_3]{1/2R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -4/5 & 6/5 & 1/5 \end{array}\right] \xrightarrow{-3R_2+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3/2 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -4/5 & 6/5 & 1/5 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} 1 & -3/2 & 0 \\ 0 & 1/2 & 0 \\ -4/5 & 6/5 & 1/5 \end{bmatrix} = \dfrac{1}{10}\begin{bmatrix} 10 & -15 & 0 \\ 0 & 5 & 0 \\ -8 & 12 & 2 \end{bmatrix}$

Rubric:

- One (1) mark for correctly setting up the augmented matrix.
- Three (3) marks for a correct reduction leading to A^{-1}.

(iii) (2 points) Find the solution to $A\mathbf{x} = \mathbf{b}$ if $\mathbf{b} = \begin{bmatrix} 5 \\ -10 \\ 10 \end{bmatrix}$.

Solution: $\mathbf{x} = A^{-1}\mathbf{b} = \dfrac{1}{10}\begin{bmatrix} 10 & -15 & 0 \\ 0 & 5 & 0 \\ -8 & 12 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -10 \\ 10 \end{bmatrix} = \dfrac{1}{10}\begin{bmatrix} 50+150 \\ -50 \\ -40-120+20 \end{bmatrix} = \begin{bmatrix} 20 \\ -5 \\ -14 \end{bmatrix}$

Rubric:

- One (1) mark for $\mathbf{x} = A^{-1}\mathbf{b}$
- One (1) mark for the correct answer.

3. Suppose that A is a 2×2 matrix. You are told that

- $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = -1$,
- $\mathbf{w} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 2$.

(i) (5 points) Write the vector $\mathbf{u} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$ as a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

Solution: We're asked to find constants x and y such that $\mathbf{u} = x\mathbf{v} + y\mathbf{w}$, or equivalently,

$$x \begin{bmatrix} 1 \\ -2 \end{bmatrix} + y \begin{bmatrix} 5 \\ -7 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \end{bmatrix},$$

which can be done by row reducing the following augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 5 & 7 \\ -2 & -7 & -5 \end{array}\right] \xrightarrow{2\mathrm{R}_1+\mathrm{R}_2\to\mathrm{R}_2} \left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 3 & 9 \end{array}\right] \xrightarrow{1/3\mathrm{R}_2\to\mathrm{R}_2} \left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{-5\mathrm{R}_2+\mathrm{R}_1\to\mathrm{R}_1} \left[\begin{array}{cc|c} 1 & 0 & -8 \\ 0 & 1 & 3 \end{array}\right].$$

Hence $x = -8$ and $y = 3$, giving $\mathbf{u} = -8\mathbf{v} + 3\mathbf{w}$.

Rubric:

- Two (2) marks for setting up the problem.
- Two (2) marks for solving the problem.
- One (1) mark for the correct answer.

(ii) (5 points) Compute $A\mathbf{u}$.

Solution: We know $A\mathbf{v} = -\mathbf{v}$ and $A\mathbf{w} = 2\mathbf{w}$, so

$$\begin{aligned} A\mathbf{u} &= A(-8\mathbf{v} + 3\mathbf{w}) = -8A\mathbf{v} + 3\mathbf{w} \\ &= -8(-\mathbf{v}) + 3(2\mathbf{w}) = 8\mathbf{v} + 6\mathbf{w} \\ &= 8 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + 6 \begin{bmatrix} 5 \\ -7 \end{bmatrix} = \begin{bmatrix} 8 + 30 \\ -16 - 42 \end{bmatrix} = \begin{bmatrix} 38 \\ -58 \end{bmatrix}. \end{aligned}$$

Rubric:

- Two (2) marks for correctly using the linear combination to solve the problem.
- Three (3) marks for the correct computation and correct answer.

4. (i) (3 points) How many arrangements of the word CHANCELLOR are there?

Solution: CHANCELLOR has 10 letters, and if we count how many times each letter occurs, we get

Letter	A	C	E	H	L	N	O	R
Count	1	2	1	1	2	1	1	1

Thus there are $\frac{10!}{2!2!}$ arrangements of this word.

Rubric:

- Two (2) marks for some argument behind why this is the case.
- One (1) mark for the correct answer.

(ii) (4 points) How many arrangements of the word CHANCELLOR have all vowels in alphabetical order (but not necessarily beside each other)?

Solution: Relative to part (i), we've over counted each arrangement $3! = 6$ times, because this is the number of ways the letters $\{A, E, O\}$ can be arranged with order mattering. Thus the number of arrangements with the vowels in alphabetical order are

$$\frac{10!/(2!2!)}{3!} = \frac{10!}{2!2!3!}.$$

Rubric:

- Three (3) marks for the argument.
- One (1) mark for the correct answer.

(iii) (3 points) How many arrangements of the word CHANCELLOR have all letters in alphabetical order?

Solution: Putting every letter into alphabetical order means that its position is uniquely determined. Thus there is precisely one, and we can write it down: ACCEHLLNOR.

Rubric:

- Two (2) marks for the argument.
- One (1) mark for the correct answer.

The formula sheet has a copy of the English alphabet, as well as the list of vowels