

University of Toronto Mississauga
Mathematical and Computational Sciences
MAT133Y5Y- Term Test 1
Test Booklet
Duration - 110 minutes
No Aids Permitted

Surname/Family Name: _____

First/Given Name: _____

Student Number: _____

UTORid: _____

Email: _____

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in. You must also have the corresponding Multiple Choice Booklet, which contains the multiple choice questions needed for Question 5 (page 6). No aids are permitted on this examination. Examples of illegal aids include, but are not limited to textbooks, notes, calculators, or any electronic device.

Each part of Question 5 includes five multiple choice answers, of which only a single response is correct. All answers should be indicated on the included scantron sheet (page 6). When filling out the scantron:

- Use a #2 pencil to bubble answers (do not use ink). A #2 pencil ensures that the marks are dark enough.
- Ensure that bubbles are completely filled and make complete erasures.

Any failure to follow these instructions which results in the scanner not being able to read your work will result in a 3 mark deduction. You will not be penalized for incorrect answers.

Your solutions to Problems 1 - 4 must be written in the space provided. No additional paper may be submitted for grading. Do not, under any circumstances, write on the QR code in the top right corner of each page. **Doing so will result in you receiving a zero (0) on this test.**

Unless otherwise indicated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work, will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

The Multiple Choice Booklet contains a **formula sheet** on its last page. Additionally, the Multiple Choice Booklet is printed one-sided, and may thus be used for rough work.

1. (i) (3 points) If f is a function, a fixed point of f is any point x_0 which satisfies $f(x_0) = x_0$. Let $f(x) = -x^5 - 10x^3 + e^x$ and let A be the set of real numbers which are fixed points for f . Write down A in terms of set builder notation. *Note:* You do not need to find the fixed points, just write down a valid set-builder formula.

Solution: No explanation is required for this question. Any statement equivalent to

$$\{x \in \mathbb{R} : -x^5 - 10x^3 + e^x = x\}$$

is correct.

Rubric: Three (3) marks for any correct answer (of which there are many possibilities).

- (ii) (7 points) Define the sets

$$A = \{-1, 0, 1\}, \quad B = \{1, 2, 3, \dots\} = \mathbb{N}, \quad C = [0, 1], \quad \text{and} \quad D = \emptyset.$$

Compute each of the following sets, and simplify your answer as much as possible. You do not need to show any work.

$$A \cap C = \underline{\hspace{10em}} \{0, 1\}$$

$$A \cap B = \underline{\hspace{10em}} \{1\}$$

$$C \cap D = \underline{\hspace{10em}} \emptyset$$

$$A \setminus C = \underline{\hspace{10em}} \{-1\}$$

$$C \setminus A = \underline{\hspace{10em}} (0, 1)$$

$$(A \cap B) \setminus C = \underline{\hspace{10em}} \emptyset$$

$$(A \cup C) \setminus B = \underline{\hspace{10em}} \{-1\} \cup [0, 1)$$

Rubric: One (1) mark for each correct answer.

2. The winner of the Cash-4-Ever Lottery receives \$1000 per week in perpetuity. For this question, assume an Annual Percentage Rate (APR) of 2%.

- (i) (2 points) In three sentences or less, explain how this company can pay \$1000 per week forever using only a finite initial principal.

Solution: The time-value of money ensures that payments made in the future are worth less than they are today, and their value continues to depreciate as the time to realization increases. As the payments occur in perpetuity, the value of those payments in the far future become insignificant, resulting in a finite initial principal.

Rubric:

- One (1) mark for discussing the time-value of money.
- One (1) mark for indicating that far-future values are effectively insignificant.

- (ii) (3 points) Derive the formula for the present value of the k th payment, where $k \geq 1$ is a natural number. *Note:* Your answer should depend on k .

Solution: The k th payment is worth \$1000 and occurs k weeks from now. This is a simple compounding interest problem, with a present value of

$$1000 \left(1 + \frac{0.02}{52} \right)^{-k}.$$

Rubric:

- One (1) mark for identifying the present value of simple compounding interest
- Two (2) marks for correctly identifying the correct values and inputting them into the correct places.

- (iii) (3 points) Write a formula for the present value of the Cash-4-Ever prize.

Solution: The effective interest rate per week is $0.02/52$, so that using our perpetuity formula we get

$$\frac{1000}{0.02/52} = \frac{52000}{0.02} = 2\,600\,000.$$

Rubric:

- One (1) mark for correctly identifying the present value of a perpetuity formula.
- Two (2) marks for correctly identifying the correct values and inputting them into the correct places.

- (iv) (2 points) Let $PV(r)$ be the present value of the Cash-4-Ever Lottery prize at an APR of $r\%$. Suppose r_1 and r_2 are positive numbers with $r_1 > r_2$. Is $PV(r_1)$ greater than $PV(r_2)$? Explain your reasoning in three sentences or less.

Solution: Mathematically, if $r_1 > r_2 > 0$ then $1/r_1 < 1/r_2$, so that

$$PV(r_1) = \frac{P}{r_1} < \frac{P}{r_2} = PV(r_2).$$

Conceptually, the present value represents the amount of money necessary to cover all future payments. If $r_1 > r_2$, then we can secure a greater interest rate with r_1 , requiring less initial principal, meaning that $PV(r_1)$ is less than $PV(r_2)$.

Rubric:

- One (1) mark for a correct answer.
- One (1) mark for a correct explanation.

3. Suppose a deposit of R dollars is made at the end of each year, and assume an APR of $r\%$ compounding *continuously*.

(i) (4 points) If we invest like this for 3 years, what is the value of the account after the third year?

Solution: At the end of each year we compound the current amount in the account and add another R dollars. We start with no money in the account, so this yields the following amounts after each year:

$$\text{Year 1: } R \quad \text{Year 2: } (R)e^r + R \quad \text{Year 3: } (Re^r + R)e^r + R = R(e^{2r} + e^r + 1).$$

Thus the account has $R(e^{2r} + e^r + 1)$ dollars after the third year.

Rubric:

- Two (2) marks for correctly computing the future value.
- One (1) mark for the correct answer (with a valid explanation).

(ii) (4 points) If we are making these payments for 3 years, how much money must we invest today to cover the payments?

Solution: Each payment must now be pulled back to the present. Again, the payments are made at the *end* of the year, so the present value of each payment is

$$\text{1st Payment: } Re^{-r}, \quad \text{2nd Payment: } Re^{-2r}, \quad \text{3rd Payment: } Re^{-3r}.$$

The total present value is the sum of these payments, yielding $Re^{-r}(e^{-2r} + e^{-r} + 1)$.

Rubric:

- Two (2) marks for correctly computing the present value.
- One (1) mark for the correct answer (with a valid explanation).

(iii) (2 points) If we invest like this for three years, but instead use an account with an APR $s\%$ compounded weekly, determine s so that this second account has the same value as your answer in part (a) after three years.

Solution: Note that it is sufficient to determine the effective rate s so that the compounding weekly yields the same return as compounding continuously in a *single year*. One can do the computation using all three years, but it's unnecessary. Setting these two numbers equal gives

$$R \left(1 + \frac{s}{52}\right)^{52} = Re^r,$$

which we solve for s to get

$$s = 52 \left(e^{r/52} - 1\right).$$

Rubric:

- One (1) mark for correctly setting up the effective rate equation.
- One (1) mark for correctly solving for the effective rate.

4. (10 points) Consider the matrix

$$A = \begin{bmatrix} 2 & 3 & 2 & 9 & -1 \\ 2 & 0 & 2 & 0 & 2 \\ 6 & 3 & 6 & 7 & -1 \\ 0 & 3 & 0 & 9 & -3 \end{bmatrix}.$$

Put A into reduced row echelon form.

Solution: This is largely a computational problem. Using Gaussian Elimination we get

$$\begin{aligned} \begin{bmatrix} 2 & 3 & 2 & 9 & -1 \\ 2 & 0 & 2 & 0 & 2 \\ 6 & 3 & 6 & 7 & -1 \\ 0 & 3 & 0 & 9 & -3 \end{bmatrix} &\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 0 & 2 & 0 & 2 \\ 2 & 3 & 2 & 9 & -1 \\ 6 & 3 & 6 & 7 & -1 \\ 0 & 3 & 0 & 9 & -3 \end{bmatrix} \xrightarrow{\substack{-R_1 + R_2 \rightarrow R_2 \\ (-3)R_1 + R_3 \rightarrow R_3}} \begin{bmatrix} 2 & 0 & 2 & 0 & 2 \\ 0 & 3 & 0 & 9 & -3 \\ 0 & 3 & 0 & 7 & -7 \\ 0 & 3 & 0 & 9 & -3 \end{bmatrix} \\ &\xrightarrow{\substack{(1/2)R_1 \rightarrow R_1 \\ -R_2 + R_3 \rightarrow R_3}} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 3 & 0 & 9 & -3 \\ 0 & 3 & 0 & 7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_2 + R_3 \rightarrow R_3} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 3 & 0 & 9 & -3 \\ 0 & 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &\xrightarrow{\substack{(1/3)R_2 \rightarrow R_2 \\ (-1/2)R_3 \rightarrow R_3}} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{(-3)R_3 + R_2 \rightarrow R_2} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Thus the reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Rubric:

- Seven (7) marks for correctly row reducing.
- Three (3) marks for correctly recognizing when to stop; for example, RREF versus REF. Give these marks even if they made computational errors, but they put the matrix in RREF.