The Robert Gillespie ACADEMIC SKILLS CENTRE

Proofs by Contradiction and Contrapositive

Proof by Contrapositive:

Proof by contrapositive can be used to prove if-then statements indirectly. The statement $P \Rightarrow Q$ is logically equivalent to its contrapositive $\neg Q \Rightarrow \neg P$. Since *logically equivalent* means that when one is true the other must also be true, if we prove that the contrapositive is true, we also proved the original statement. Hint: use the contrapositive when there are “negative” statements (no solutions, not natural, $\neq$, $\notin$, etc.), when the if part is hard to start from, or simply when stuck.

Example. Use the contrapositive the prove that if $c > \frac{49}{8}$ then $f(x) = 2x^2 + 7x + c$ has no solutions.

Solution.
First, we formulate the contrapositive:

- P is $c > \frac{49}{8}$ so $\neg P$ is $c \leq \frac{49}{8}$.
- Q is $f(x) = 2x^2 + 7x + c$ has no solutions so $\neg Q$ is $f(x) = 2x^2 + 7x + c$ has at least one solution.

Therefore, the contrapositive is if $f(x) = 2x^2 + 7x + c$ has at least one solution then $c \leq \frac{49}{8}$. For a quadratic polynomial $f(x) = ax^2 + bx + c$ to have roots its determinant must satisfy $b^2 - 4ac \geq 0$. For our polynomial that is $(7)^2 - 4(2)(c) \geq 0$. We try to isolate c:

$$\begin{aligned} (7)^2 - 4(2)(c) &\geq 0 \\ 49 - 8c &\geq 0 \\ 49 &\geq 8c \\ \frac{49}{8} &\geq c \end{aligned}$$

which was what we wanted. Since the contrapositive is equivalent to the original statement, we have proven that if $c > \frac{49}{8}$ then $f(x) = 2x^2 + 7x + c$ has no solutions. ◆

Proof by Contradiction:

Proof by contradiction can be used to prove any statement, including if-then. To prove a statement P by contradiction:

1. Formulate $\neg P$ and assume it is true.
2. Manipulate the statement(s) in $\neg P$ to find a contradiction. A contradiction means two opposing statements being true at the same time (e.g. $x = 0$ and $x \neq 0$).

Hint: use contradiction when there are negative statements, "or" statements or simply when stuck.

Example. Prove that if $a^2 + 5$ is even then a is odd.

Solution.

1. The statement to prove in this case has the form $P \Rightarrow Q$, where P means $a^2 + 5$ is even and Q means a is odd. The negation of $P \Rightarrow Q$ is $P \wedge \neg Q$, that would be $a^2 + 5$ is even and a is even.

2. Now, we must assume these facts are true and use them to achieve a contradiction.

We've assumed that a is even, that means $a = 2k$ for some $k \in \mathbb{Z}$. We can use this information about a and plug it into $a^2 + 5$:

$$a^2 + 5 = (2k)^2 + 5$$
$$a^2 + 5 = 4k^2 + 5$$
$$a^2 + 5 = 4k^2 + 4 + 1$$
$$a^2 + 5 = 2(2k^2 + 2) + 1$$

Since $a^2 + 5$ takes the form $2m + 1$ for $m \in \mathbb{Z}$, it must be odd. However, we originally said $a^2 + 5$ is even. This is a contradiction; therefore the negation must be false and the original statement must be true. ◆