Mathematical Induction and Strong Induction

Induction:

Use induction when you are asked to prove a statement involving a natural number $P(n)$, for all natural numbers (e.g. Show $2n-1$ is odd for all $n \in \mathbb{N}$).

There are two parts to induction: Base Case and Induction Step

1. In the base case, we show $P(n)$ true for a starting value, usually $P(1)$.
2. In the induction, step we show that if $P(n)$ works for a certain number k it will also work for the next number $k+1$ (you prove $P(k) \Rightarrow P(k+1)$). That is, you assume $P(k)$ is true and use it to show that $P(k+1)$ is also true. The statement $P(k)$ is called the **induction hypothesis**.

Example. Prove that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all $n \in \mathbb{N}$.

Solution.
Since we are being asked to prove something for all natural numbers, we will use induction. Here $P(n)$ means $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

1. Base Case:

$P(1)$ means $\sum_{i=1}^{1} i^3 = \left(\frac{1(1+1)}{2}\right)^2$; check that the two sides are equal:

$\sum_{i=1}^{1} i^3 = 1^3 = 1$ and $\left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1$. Therefore, $P(1)$ is true.

2. Induction Step:

Induction Hypothesis:

Assume $P(k)$ is true, that is $\sum_{i=1}^{k} i^3 = \left(\frac{k(k+1)}{2}\right)^2$. We will be using this fact in our proof.

Now, we must prove $P(k+1)$, that is $\sum_{i=1}^{k+1} i^3 = \left(\frac{(k+1)((k+1)+1)}{2}\right)^2$.

We will start with the left side and show that it is equal to the right side. First expand the left side:

$$\sum_{i=1}^{k+1} i^3 = 1^3 + \ldots + k^3 + (k+1)^3$$

$$= \sum_{i=1}^{k} i^3 + (k+1)^3$$

Notice that $\sum_{i=1}^{k} i^3$ is a part of our hypothesis, we replace it with $\left(\frac{k(k+1)}{2}\right)^2$.

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

We use algebra to show that this is equal to the right side:

$$\left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = (k+1)^2\left(\frac{k^2}{4}+k+1\right)$$
$$= \frac{(k+1)^2}{4}(k^2+4k+4)$$
$$= \frac{(k+1)^2(k+2)^2}{4}$$

We are done. Since we were able to show $P(k) \Rightarrow P(k+1)$ and our base case, we conclude that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all $n \in \mathbb{N}$. ◆

Strong Induction:

There are some statements $P(n)$ about the natural numbers which regular induction cannot prove, in which case we must use Strong Induction. The main difference is that instead of using *only* $P(k)$ to prove $P(k+1)$ we are allowed to use *all* of $P(1),\ldots,P(k)$ to prove it. We may not need all of them, but we assume they are all true.

We still have the same two steps, but they are slightly altered:

1. In the base case we prove more cases, the number depends on the question. Showing 3 base cases will be enough for most, usually $P(1), P(2)$ and $P(3)$.
2. In the induction step we assume $P(1),\ldots,P(k)$ are true and use them to prove $P(k+1)$. That is, we prove $P(1),\ldots,P(k) \Rightarrow P(k+1)$.

Example. Assume that $a_{n+1} = 6a_{n-1} - 12a_n + 3$ with $a_1 = 1$ and $a_2 = 3$. Show that a_n is odd for all $n \in \mathbb{N}$.

Solution.
In this case $P(n)$ is the statement a_n is odd.

1. Base Case

- $P(1)$ means a_1 is odd; since $a_1 = 1$ we know it is odd.
- Similarly, $a_2 = 3$ is odd so $P(2)$ is true.
- To check a_3 we must use the recursive formula:

$$a_3 = 6a_2 - 12a_1 + 3$$
$$a_3 = 6(3) - 12(1) + 3$$
$$a_3 = 9$$

So a_3 is odd and $P(3)$ is true.

2. Induction Step:

Induction Hypothesis:
Assume that for some $k \in \mathbb{N}$ we have that $P(1), P(2), ..., P(k-1), P(k)$ are all true. That is, $a_1, a_2, ..., a_{k-1}, a_k$ are all odd. We will be using this fact in our proof.

Now, we must prove $P(k+1)$ is true. That is, a_{k+1} is odd. To talk about a_{k+1} we must use the recursive formula: $a_{k+1} = 6a_{k-1} - 12a_k + 3$.
What do we know about a_{k-1} and a_k? The induction hypothesis says they are odd! We can write them as $a_{k-1} = 2x + 1$ and $a_k = 2y + 1$ with $x, y \in \mathbb{Z}$. Let's use these in the formula and try to show a_{k+1} is odd. We are looking to express a_{k+1} as $a_{k+1} = 2(\sim) + 1$ where $\sim$ is an integer:

$$a_{k+1} = 6(2x+1) - 12(2y+1) + 3$$
$$a_{k+1} = 6(2x+1) - 12(2y+1) + 2 + 1$$
$$a_{k+1} = 2\left[3(2x+1) - 6(2y+1) + 1\right] + 1$$

Since $3(2x+1) - 6(2y+1) + 1$ is an integer, a_{k+1} must be odd.
Therefore by strong induction a_n is odd for all $n \in \mathbb{N}$. ◆