Logical Equivalence and Negating Logical Statements

Two logical statements are equivalent if the values in their truth tables are all equal.

Example. Show $\neg(P \Rightarrow Q)$ is equivalent to $P \wedge \neg Q$.

Solution.
To make the truth table for logical statement(s) we break them down into their constituent parts. For example $\neg(P \Rightarrow Q)$ is made up from P and Q, which are put together as $P \Rightarrow Q$ and finally negated into $\neg(P \Rightarrow Q)$.

In $P \wedge \neg Q$ we still have P and Q, but we also need $\neg Q$ which together with P makes $P \wedge \neg Q$.

First we fill in the columns for the statements P and Q, making sure we have every combination of true and false (i.e., every combination of their values).

P	Q	$\neg Q$	$P \wedge \neg Q$	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$
T	T				
T	F				
F	T				
F	F				

Then we fill in the rest of the columns using the tables for the other connectives (not, and, or, …).

P	Q	$\neg Q$	$P \wedge \neg Q$	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$
T	T	F	F	T	F
T	F	T	T	F	T
F	T	F	F	T	F
F	F	T	F	T	F

Since the columns for $\neg(P \Rightarrow Q)$ and $P \wedge \neg Q$ are the same, the statements are logically equivalent. ◆

Negating Statements

We negate a statement by using the following formulas/rules:

1. $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
2. $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
3. $\neg(P \Rightarrow Q) \Leftrightarrow P \wedge \neg Q$
4. $\neg((\forall x) R(x)) \Leftrightarrow (\exists x) \neg R(x)$
5. $\neg((\exists x) R(x)) \Leftrightarrow (\forall x) \neg R(x)$

Note that in negation, we switch *and* ($\wedge$) and *or* ($\vee$) and *for every* ($\forall$) and *there exists* ($\exists$).

Example. Show the negation of

$$(\forall x \in F)\left[x \neq 0 \Rightarrow (\exists y \in F)(y \neq 0 \wedge x \cdot y = 1)\right]$$

and simplify so that the formula does not include the negation symbol $\neg$.

Solution.

$$\neg\left\{(\forall x \in F)\left[x \neq 0 \Rightarrow \left((\exists y \in F)(y \neq 0 \wedge x \cdot y = 1)\right)\right]\right\}$$
$$(\exists x \in F)\neg\left[x \neq 0 \Rightarrow \left((\exists y \in F)(y \neq 0 \wedge x \cdot y = 1)\right)\right]$$
$$(\exists x \in F)\left[x \neq 0 \wedge \neg\left((\exists y \in F)(y \neq 0 \wedge x \cdot y = 1)\right)\right]$$
$$(\exists x \in F)\left[x \neq 0 \wedge \left((\forall y \in F)\neg(y \neq 0 \wedge x \cdot y = 1)\right)\right]$$
$$(\exists x \in F)\left[x \neq 0 \wedge \left((\forall y \in F)(y = 0 \vee x \cdot y \neq 1)\right)\right]$$

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Example. Compute the negative of $(\exists x \in \mathbb{R})\left(\left((\forall y \in \mathbb{R})(y^2 \neq x)\right) \vee (x \geq 0)\right)$ without using the $\neg$ symbol.

Solution.

$$\neg\left\{(\exists x \in \mathbb{R})\left(\left((\forall y \in \mathbb{R})(y^2 \neq x)\right) \vee (x \geq 0)\right)\right\}$$
$$(\forall x \in \mathbb{R})\neg\left(\left((\forall y \in \mathbb{R})(y^2 \neq x)\right) \vee (x \geq 0)\right)$$
$$(\forall x \in \mathbb{R})\left(\neg\left((\forall y \in \mathbb{R})(y^2 \neq x)\right) \wedge \neg(x \geq 0)\right)$$
$$(\forall x \in \mathbb{R})\left(\left((\exists y \in \mathbb{R})\neg(y^2 \neq x)\right) \wedge (x < 0)\right)$$
$$(\forall x \in \mathbb{R})\left(\left((\exists y \in \mathbb{R})(y^2 = x)\right) \wedge (x < 0)\right)$$

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