TheRobert Gillespie ACADEMIC SKILLS CENTRE

Image of a function

The image of a function $f : A \to B$ is $f(A) = \{f(a) : a \in A\}$.
To find the image of a function we must find all the outputs obtained from all the inputs in the domain. It can help to input different values into the function and seeing the outputs (e.g. close to zero, positives, negatives, very large numbers, etc.). The image of a function is sometimes referred to as the range of a function. More formally, we start with an arbitrary element in the domain and any conditions on it, then construct the function and any changes to those conditions.

Example. Prove that the image of the function $f : [0,\infty) \to \mathbb{R}, f(x) = \dfrac{5x}{x+7}$ is $f([0,\infty)) = [0,5)$.

Solution.
Since $f([0,\infty))$ and $[0,5)$ are sets we must show both $f([0,\infty)) \subseteq [0,5)$ and $[0,5) \subseteq f([0,\infty))$.

We will first show that $f([0,\infty)) \subseteq [0,5)$. Pick an x in $[0,\infty)$, which means that $x \geq 0$. Then $f(x) = \dfrac{5x}{x+7} \geq 0$ because both the numerator and the denominator are positive (and the numerator could be zero). As well, when $x > 0$ then $f(x) = \dfrac{5x}{x+7} < \dfrac{5x}{x} = 5$, and putting the two inequalities together, we get $0 \leq f(x) < 5$ i.e., $f([0,\infty)) \subseteq [0,5)$.

Now, we must show that $[0,5) \subseteq f([0,\infty))$. This means that, given any $y \in [0,5)$, we have to find a number $a \in [0,\infty)$ such that $y = f(a)$. Expanding $y = f(a)$ and solving for a we get:

$$y = \frac{5a}{a+7}$$

$$ay + 7y = 5a$$

$$a(y-5) = -7y$$

$$a = \frac{-7y}{y-5} = \frac{7y}{5-y}$$

Since $y \in [0,5)$, the numerator is positive or zero, and the denominator is positive. Thus, $a \geq 0$, i.e., $a \in [0,\infty)$, as required. ◆