Bijections

Injection:

A function $f : A \to B$ is an injection when different inputs are always mapped to different outputs, that is, $x \neq y \Rightarrow f(x) \neq f(y)$.

In practice we normally use the contrapositive ($f(x) = f(y) \Rightarrow x = y$) as it is generally easier to prove.

To show a function is *not* injective, we must show a counterexample. We would need two different inputs $x_1 \neq x_2$ so that their outputs are the same $f(x_1) = f(x_2)$.

Surjection:

A function $f : A \to B$ is a surjection if every element of the codomain B is mapped onto. In other words, for any $b \in B$ we can find at least one $a \in A$ so that $f(a) = b$. One way to do this is to:

1. Pick an arbitrary element of the codomain (let $b \in B$, without specifying anything else about it).
2. Look for $a \in A$ so that $f(a) = b$ (do this by plugging in x into f, set $f(x) = b$ and solve for x in terms of b).
3. Confirm that plugging in this x into f actually does output b, as well as making sure that $x \in A$.

Another way is to prove that the image of f is equal to the codomain, i.e.,

$$im(f) = B$$

To show a function is not a surjection, we again must find a counterexample. That would be finding $b \in B$ so that it isn't possible that $f(a) = b$ for any $a \in A$.

Bijection:

A bijection is a function $f : A \to B$ which is both **injective** and **surjective**. Bijections can be reversed, if f is a bijection then there is a function $f^{-1} : B \to A$ (called the inverse function of f) which undoes what f does.

Formally,

$$f(a) = b \text{ if and only if } f^{-1}(b) = a$$

This is demonstrated in the fact that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. It is useful to note that f^{-1} is also a bijection.

Example. Determine whether the following functions are injective, surjective, bijective or neither.

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} -\dfrac{1}{x}; & x < 0 \\ -x^2 + 1; & x \geq 0 \end{cases}$

(b) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(a,b) = a \cdot b$

Solution Part (a).
Since we have a piecewise function, we must handle it slightly differently.

For surjectivity, we must show the combined images of each part are equal to the codomain $\mathbb{R}$. For injectivity, we must show that each function is injective *as well as show that* $-\frac{1}{x_1} \neq -x_2{}^2 + 1$, *that is,* $f(x_1) \neq f(x_2)$.

<u>We will test surjection first:</u>

First, take $x < 0$, so $f(x) = -\frac{1}{x}$. We want to build up $-\frac{1}{x}$ within the $x < 0$ inequality:

$$x < 0 \Rightarrow \frac{1}{x} < 0 \Rightarrow -\frac{1}{x} > 0 \Rightarrow f(x) > 0$$

Therefore, the image of f when $x < 0$ is $(0, \infty)$.

Now for the second part, we begin with $x \geq 0$ and build up $-x^2 + 1$ within the inequality:

$$x \geq 0 \Rightarrow x^2 \geq 0 \Rightarrow -x^2 \leq 0 \Rightarrow -x^2 + 1 \leq 1$$

Thus, the image of f when $x \geq 0$ is $(-\infty, 1]$.

Putting these two together, we see that the image of f is:

$$(0, \infty) \cup (-\infty, 1] = (-\infty, \infty) = \mathbb{R}$$

and f is surjective.

<u>We now test injection:</u>

From the work on surjectivity above we see that the images of the two functions that make up f overlap. This means we should be able to find $x_1 < 0$ and $x_2 \geq 0$ so that $f(x_1) = f(x_2)$, that is, $-\frac{1}{x_1} = -x_2{}^2 + 1$. One way to do this is to try guessing using "easy" numbers. For instance, when $x_2 = 0$ then $-\frac{1}{x_1} = -x_2{}^2 + 1$ implies $-\frac{1}{x_1} = 1$ and $x_1 = -1$. Done! As $f(0) = 1$ and

$f(-1)=1$, we see that $f(0)=f(-1)$, however, $0\neq -1$ so f is not injective.

Alternatively, we have to try to find these x_1 and x_2 to prove the function is not injective. It seems easier to solve for x_1:

$$-\frac{1}{x_1}=-x_2{}^2+1$$

$$-1=(-x_2{}^2+1)x_1$$

$$\frac{-1}{-x_2{}^2+1}=x_1$$

Since we said $x_1<0$, we can apply this to our expression: $\frac{-1}{-x_2{}^2+1}<0$. Since the numerator is a negative, the denominator must be a positive for this to be true. That is, $-x_2{}^2+1>0$. We can then solve for x_2:

$$-x_2{}^2+1>0\Rightarrow -x_2{}^2>-1\Rightarrow x_2{}^2<1\Rightarrow |x_2|<1\Rightarrow -1<x_2<1$$

Since $x_2\geq 0$ this is restricted to $0\leq x_2<1$. We pick a value from that range, the easiest being $x_2=0$ and find the respective x_1: $\frac{-1}{0^2+1}=x_1\Rightarrow x_1=-1$

From here we check their values: $f(0)=1$ and $f(-1)=1$. So $f(0)=f(-1)$, however, $0\neq -1$ so f is not injective.

Since f is not injective, it is not bijective. Thus f is only surjective.

Solution Part (b).

<u>We test injection:</u>

One way is to ask the following question: can we find two different pairs of numbers, which, when multiplied, give the same number? Start, for instance, with $f(2,4)=2\cdot 4=8$. Now we need a different pair whose product is 8; there are many such pairs, for instance, $f(1,8)=1\cdot 8=8$. Thus, f is not injective.

Alternatively: we would like to prove that for $(a,b),(c,d)\in\mathbb{R}$ we have:

$$f(a,b)=f(c,d)\Rightarrow(a,b)=(c,d)$$

i.e., that $a=c$ and $b=d$. So, we will try to prove it. We assume $f(a,b)=f(c,d)$, this would mean $ab=cd$. This doesn't give us much, as we cannot conclude that $a=c$ and $b=d$. Therefore, we look for a counterexample:

$$f(2,6)=2\cdot 6=12 \text{ and } f(3,4)=3\cdot 4=12.$$

Meanwhile, $(2,6)\neq(3,4)$. Therefore, f is not injective.

<u>Testing surjection:</u>

We pick an arbitrary element of the codomain, let $r\in\mathbb{R}$. We must find $(a,b)\in\mathbb{R}^2$ such that $f(a,b)=r$, so we expand the function: $ab=r$. It will be hard to find two variables in one equation so we "fix" one of them to make our work easier; let $b=1$ and we try to find a to make the equation work:

$$\begin{aligned} ab&=r\\ a(1)&=r\\ a&=r \end{aligned}$$

This tells us $(1,r)$ satisfies our requirement, we confirm it: $f(1,r)=1\cdot r=r$.

Since we were able to find $(a,b)\in\mathbb{R}^2$ so that $f(a,b)=r$ for arbitrary $r\in\mathbb{R}$, f is a surjective function.

Because it is not injective, f is not bijective. ◆