Matrix Inverse

The inverse of an $n \times n$ matrix A (if it exists) is defined to be the matrix B so that $BA = I$ and $AB = I$, where I is the identity matrix. If the inverse exists, A is called invertible and its inverse is denoted by A^{-1}.

To determine if an $n \times n$ matrix A is invertible, we can use the determinant:

An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

For an invertible matrix A, the system of linear equations $Ax = b$ has the unique solution $x = A^{-1}b$.

<u>Computing Inverses:</u>

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have a formula we can use to find the inverse:

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

For a more general $n \times n$ invertible matrix A, we use the following steps:

1. Augment A with the identity matrix (I) of the same size.

2. Use row operations to take A to reduced row echelon form while performing those operations on the augmented matrix, i.e., including I as well. This should result in I in the place where A was.
3. Once the original A becomes I, the matrix in the place of where I was is the inverse A^{-1}.

Here is a visualization: $\left[A \middle| I\right] \rightarrow \ldots \rightarrow \left[I \middle| A^{-1}\right]$.

<u>Useful Facts:</u>

Let A and B be invertible matrices.

(1) $I^{-1} = I$

(2) $(A^{-1})^{-1} = A$

(3) $(AB)^{-1} = B^{-1}A^{-1}$

(4) $(A^k)^{-1} = (A^{-1})^k$, where $k \in$

(5) $(aA)^{-1} = \frac{1}{a}A^{-1}$, where $a \neq 0$ is a scalar

(6) $(A^T)^{-1} = (A^{-1})^T$, where T denotes the transpose of a matrix

Example. Find the inverses of the following matrices.

(a) $A = \begin{bmatrix} 8 & 7 \\ 1 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 3 & 7 & 2 \\ 5 & 2 & 6 \\ 1 & 0 & 4 \end{bmatrix}$

Solution Part (a).
First, we'll find out if it is invertible by finding the determinant.

$$\det(A) = (8)(2) - (1)(7) = 9 \neq 0$$

Since the determinant is non-zero, we use the formula for a 2×2 matrix:

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} 2 & -7 \\ -1 & 8 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 2 & -7 \\ -3 & 8 \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & \frac{-7}{9} \\ \frac{-3}{9} & \frac{8}{9} \end{bmatrix}$$

Solution Part (b).
Again, find if A is invertible first by computing its determinant:

$$\begin{aligned} \det(A) &= 1(42-4) - 0 + 4(6-35) \\ &= 38 + 4(-29) \\ &= -78 \neq 0 \end{aligned}$$

Apply the row reduction to A augmented by I:

$$\left[\begin{array}{ccc|ccc} 3 & 7 & 2 & 1 & 0 & 0 \\ 5 & 2 & 6 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R1 \leftrightarrow R3$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 0 & 1 \\ 5 & 2 & 6 & 0 & 1 & 0 \\ 3 & 7 & 2 & 1 & 0 & 0 \end{array}\right]$$

$R2 \leftarrow R2 - 5R1$
$R3 \leftarrow R3 - 3R1$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 0 & 1 \\ 0 & 2 & -14 & 0 & 1 & -5 \\ 0 & 7 & -10 & 1 & 0 & -3 \end{array}\right]$$

$R2 \leftarrow \frac{1}{2}R2$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 0 & 1 \\ 0 & 1 & -7 & 0 & \frac{1}{2} & \frac{-5}{2} \\ 0 & 7 & -10 & 1 & 0 & -3 \end{array}\right]$$

$$R3 \leftarrow R3 - 7R2 \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 0 & 1 \\ 0 & 1 & -7 & 0 & \frac{1}{2} & \frac{-5}{2} \\ 0 & 0 & 39 & 1 & \frac{-7}{2} & \frac{29}{2} \end{array}\right]$$

$$R3 \leftarrow \frac{1}{39} R3 \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 0 & 1 \\ 0 & 1 & -7 & 0 & \frac{1}{2} & \frac{-5}{2} \\ 0 & 0 & 1 & \frac{1}{39} & \frac{-7}{78} & \frac{29}{78} \end{array}\right]$$

$$\begin{array}{l} R1 \leftarrow R1 - 4R3 \\ R2 \leftarrow R2 + 7R3 \end{array} \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-4}{39} & \frac{14}{39} & \frac{-19}{39} \\ 0 & 1 & 0 & \frac{7}{39} & \frac{-5}{39} & \frac{4}{39} \\ 0 & 0 & 1 & \frac{1}{39} & \frac{-7}{78} & \frac{29}{78} \end{array}\right]$$

Since we finally made it to the identity matrix on the left side, the right side is our inverse. We can factor out a $\frac{1}{39}$ from all the entries to make it easier to read:

$$A^{-1} = \frac{1}{39}\begin{bmatrix} -4 & 14 & -19 \\ 7 & -5 & 4 \\ 1 & \frac{-7}{2} & \frac{29}{2} \end{bmatrix}$$

◆