Matrix Row Reduction

We usually perform row reduction on matrices to solve systems of equations. It can also be used to find inverses. We're allowed to use the following elementary row operations on a matrix in order to reduce it:

Elementary Row Operations

1. Interchanging rows
2. Multiplying one row by a nonzero number
3. Adding a multiple of one row to a different row

The last two are usually combined to make calculations easier. For example, $R2 \leftarrow 3R2 + 2R1$ signifies that Row 2 will become 3 times Row 2 plus 2 times Row 1. This can be separated into multiplying Row 2 by 3 then adding to it Row 1 multiplied by 2, but it is easier to combine them.

Generally, we aim to reduce the matrix to **row echelon form**. This means each of the leftmost entries (the leading term) in a row is a 1 and each entry below the leading 1 is a 0.

The general steps to get a matrix to row echelon form are as follows:

1. Find a row in the leftmost column with a non-zero entry (a) and move it to the top row.
2. Multiply the row by $1/a$ to obtain a leading 1.
3. Use the leading 1 to make the entries in the rows below into zeroes.
4. Repeat these steps, moving one column to the right.

For **reduced row echelon form**, use the leading 1s to transform the entries above them into 0s.

Example. Transform $\left[\begin{array}{ccc|c} 2 & -8 & 8 & 4 \\ 1 & -3 & 1 & 7 \\ -6 & 3 & -15 & -9 \end{array}\right]$ into reduced row echelon form.

Solution.
We'll follow the steps outlined above to reduce the matrix. There is already a row with a leading 1 so we will move it to the top row. Then we use the leading 1 to eliminate the entries below it. Multiply Row 1 by -2, then add it to Row 2 to get rid of the 2. A similar calculation gives a 0 in Row 3:

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 7 \\ 2 & -8 & 8 & 4 \\ -6 & 3 & -15 & -9 \end{array}\right] \begin{array}{l} R2 \leftarrow R2 - 2R1 \\ R3 \leftarrow R3 + 6R1 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 7 \\ 0 & -2 & 6 & -10 \\ 0 & -15 & -9 & 33 \end{array}\right]$$

Now, that we've gotten rid of the entries below our leading 1, we can move on to the next row. We want a leading 1 for the second row, so we divide the whole row by -2 ($R2 \leftarrow \frac{-1}{2} R2$).

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 7 \\ 0 & 1 & -3 & 5 \\ 0 & -15 & -9 & 33 \end{array}\right]$$

Now, we can remove the entry below our leading 1 by multiplying by 15 and adding to Row 3 ($R3 \leftarrow R3 + 15R2$). Notice the 0s in the first column are unaffected. Finally, make the remaining entry in Row-3 into a 1:

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 7 \\ 0 & 1 & -3 & 5 \\ 0 & -15 & -9 & 33 \end{array}\right] R3 \leftarrow \frac{-1}{54} R3 \left[\begin{array}{ccc|c} 1 & -3 & 1 & 7 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & -54 & 108 \end{array}\right]$$

Now, our matrix is in row echelon form. Next we get rid of the entries above the leading 1s. Divide Row 3 by -54. Begin the process with Column 3 to avoid affecting other rows due to the 0s, then move on to Column 2:

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 7 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{l} R2 \leftarrow R2 + 3R3 \\ R1 \leftarrow R1 - R3 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 9 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Finally,

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 9 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right] R1 \leftarrow R1 + 3R2 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Note that in this example we had an extra column. Thus, we were actually solving a system of linear equations, and the final matrix given the values of the three unknowns. ◆