The Robert Gillespie ACADEMIC SKILLS CENTRE

Determinant of a Matrix

The determinant of a 2×2 matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A)=ad-bc$.

For example, $\det\left(\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}\right)=(1)(-4)-(3)(-2)=-4+6=2$.

In general, for an $n\times n$ matrix A with $n\geq 2$, we use the **cofactor expansion**:

1. Choose a column or row, preferably the one with the most 1s and 0s.
2. For the entry a_{ij}, let A_{ij} denote the matrix obtained by deleting row i and column j.
3. Calculate $c_{ij}(A)=(-1)^{i+j}\det(A_{ij})$, this is called the (i,j)-cofactor. You can use this diagram to find $(-1)^{i+j}$ for each entry:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

4. Repeat steps 2 and 3 for each entry in the row/column you chose.
5. Multiply each cofactor by its corresponding entry, then add them all together. Here is the determinant of A using the first row:

$$\det(A)=a_{11}c_{11}(A)+a_{12}c_{12}(A)+a_{13}c_{13}(A)+\ldots+a_{1n}c_{1n}(A)$$

Keep in mind if any of the $a_{11},\ldots,a_{1n}$ is 0, then the corresponding term $a_{ij}c_{ij}(A)$ disappears, so there is no need to calculate the cofactor!

For example, the determinant of a 3×3 matrix $B=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ using Row 1:

$$\det(B) = a\det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b\det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c\det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$
$$= a(ei-fh) - b(di-fg) + c(dh-eg)$$

Example. Find the determinant of the following matrices.

(a) $M=\begin{bmatrix} 1 & 4 & 1 \\ 2 & 7 & 1 \\ 3 & 1 & 4 \end{bmatrix}$ (b) $M=\begin{bmatrix} 9 & 9 & 4 \\ 0 & 2 & 0 \\ 1 & 3 & 2 \end{bmatrix}$

Solution Part (a).
We will find the determinant using Column 3, since it has two 1s. Here are the calculations for the cofactors:

$$c_{13}(M) = (-1)^{1+3}\det\left(\begin{bmatrix} 2 & 7 \\ 3 & 1 \end{bmatrix}\right) = (1)(2-21) = -19$$
$$c_{23}(M) = (-1)^{2+3}\det\left(\begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}\right) = (-1)(1-12) = 11$$
$$c_{33}(M) = (-1)^{3+3}\det\left(\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}\right) = (1)(7-8) = -1$$

Now, we can calculate the determinant:

$$\begin{aligned}\det(M) &= a_{13}c_{13}(M) + a_{23}c_{23}(M) + a_{33}c_{33}(M) \\ &= (1)(-19) + (1)(11) + (4)(-1) \\ &= -9\end{aligned}$$

Solution Part (b).

Since Row 2 has two 0s we will use that row, it will make the calculation much easier. We will only calculate the $(2,2)$-cofactor since the other two aren't needed:

$$c_{22}(M) = (-1)^{2+2}\det\left(\begin{bmatrix} 9 & 4 \\ 1 & 2 \end{bmatrix}\right) = (1)(18-4) = 14$$

Calculating the determinant, we get:

$$\begin{aligned}\det(M) &= (0)c_{21}(M) + (2)c_{22}(M) + (0)c_{23}(M) \\ &= (0)(0) + (2)(14) + (0)(0) = 28\end{aligned}$$

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