The Robert Gillespie ACADEMIC SKILLS CENTRE

Partial Derivatives, Gradient, Divergence, and Curl

Partial Derivatives

Recall that the derivative of a function of one variable provides information about how the function changes as the input variable changes. For a function of multiple variables, we consider how the function changes when one input variable changes and all other input variables are fixed, and refer to partial derivatives. For example, if f is a function of two variables x and y, then its partial derivatives are the functions f_x and f_y, which are defined by

$$f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} \text{ and } f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

provided that the limit(s) exist.

Notation for Partial Derivatives

For a function $z=f(x,y)$, all of the following notation is used to represent partial derivatives.

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Rule for Finding Partial Derivatives of $z=f(x,y)$

1. To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

Partial derivatives are similarly defined for functions of three or more variables.

Higher Order Derivatives

For higher order derivatives, we take derivatives one order at a time. For a function $z = f(x, y)$, we use the notation

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

The notation is similar for higher order partial derivatives.

Clairaut's Theorem

Suppose that f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$. In essence, if f is a "nice" function, then a mixed partial derivative is independent of the order we take the derivatives, i.e. $f_{xy} = f_{yx}$.

Del Operator

The del operator is denoted as ∇ which is called nabla. The del operator is a vector written as

$$\nabla = \left(\frac{\partial}{\partial x}\right)\bar{i} + \left(\frac{\partial}{\partial y}\right)\bar{j} + \left(\frac{\partial}{\partial z}\right)\bar{k}$$

Basically, the presence of the ∇ tells us we are taking a derivate, which takes several forms, each of which gives different results!

- If f is a function, then ∇f (gradient) is a vector.

- If $\bar{F}$ is a vector field, then $\nabla \cdot \bar{F}$ (divergence) is a scalar function.
- If $\bar{F}$ is a vector field, then $\nabla \times \bar{F}$ (curl) is a vector function.

We will consider each of these forms below.

Gradient

If f is a function of two variables x and y, then the gradient of f is the vector ∇f defined by

$$\operatorname{grad} f(x,y) = \nabla f(x,y) = \left\langle f_x(x,y), f_y(x,y) \right\rangle = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j}$$

where $\bar{i}$ and $\bar{j}$ are unit vectors.

The gradient of a function is similarly defined for functions of three or more variables.

We think of the gradient of a function of two or more variables as the direction of the fastest increase of f at a point, with magnitude equal to the maximum rate of increase at a point – just as we do for the derivative of a single-variable function. Geometrically, the gradient at a point is perpendicular to the tangent vector of any curve that passes through the point.

Example. If $f(x,y) = 2x^2y^3 - 4y + 5$, find the gradient at the point $(2,-1)$.

Solution. Taking the required partial derivatives, we obtain

$$\nabla f(x,y) = \left\langle f_x(x,y),\ f_y(x,y) \right\rangle = \left\langle 4xy^3, 6x^2y^2 - 4 \right\rangle.$$

Plugging in the point $(2,-1)$, we obtain

$$\nabla f(2,-1) = \left\langle 4(2)(-1)^3, 6(2)^2(-1)^2 - 4 \right\rangle = \langle -8, 20 \rangle.$$

Divergence

If $\bar{F} = P\bar{i} + Q\bar{j} + R\bar{k}$, where P, Q, and R are functions, is a vector field on $\mathbb{R}^3$, and $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$ exist, then the divergence of $\bar{F}$ is defined by

$$\nabla \cdot \bar{F} = \operatorname{div} \bar{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

The divergence of a function is similarly defined for functions of an arbitrary number of variables.

In essence, the divergence is the dot product of the del operator ∇ and the vector field $\overline{F}$ which results in a scalar function.

Example. If $\overline{F} = xz\overline{i} + xyz\overline{j}$, then find $\nabla \cdot \overline{F}$.

Solution. We note that $R = 0$. Applying the definition, we obtain

$$\nabla \cdot \overline{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(0) = z + xz.$$

Curl

If $\overline{F} = P\overline{i} + Q\overline{j} + R\overline{k}$, where P, Q, and R are functions, is a vector field on R^3, and $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$ exist, then the curl of $\overline{F}$ is defined by

$$\nabla \times \overline{F} = \operatorname{curl} \overline{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\overline{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\overline{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\overline{k}$$

Or equivalently, $\nabla \times \overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$.

The curl of a function is **not** defined for functions with a different number of variables; it only applies to 3-dimensional space.

In essence, the curl is the cross product of the del operator ∇ and the vector field $\overline{F}$ which results in a vector field.

Geometrically, the curl of $\overline{F}$ is a vector that is perpendicular to the vector field that represents the velocity of a particle in motion. Its length and direction characterize the rotation at a particular point.

Example. If $\overline{F} = xyz^2\overline{i} - x^2y\overline{j} + x^2yz^3\overline{k}$, then find $\nabla \times \overline{F}$.

Solution. Applying the definition, we obtain

$$
\begin{aligned}
\nabla \times \overline{F} &= \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz^2 & -x^2 y & x^2 yz^3 \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y} x^2 yz^3 - \frac{\partial}{\partial z}\left(-x^2 y\right) \right)\overline{i} + \left(\frac{\partial}{\partial z} xyz^2 - \frac{\partial}{\partial x} x^2 yz^3 \right)\overline{j} + \left(\frac{\partial}{\partial x}(-x^2 y) - \frac{\partial}{\partial y} xyz^2 \right)\overline{k} \\
&= \left(x^2 z^3\right)\overline{i} + \left(2xyz - 2xyz^3\right)\overline{j} + \left(-2xy - xz^2\right)\overline{k}.
\end{aligned}
$$

Fun Facts:

Theorem. If f is a function of three variables that has continuous second-order partial derivatives, then

$$\nabla \times \nabla f = \operatorname{curl}\left(\nabla f\right) = \overline{0}.$$

It follows that if $\overline{F}$ is conservative, meaning there is a function f such that $\overline{F} = \nabla f$, then

$$\nabla \times \overline{F} = \operatorname{curl}\left(\overline{F}\right) = \overline{0}.$$

Theorem. If $\overline{F} = P\overline{i} + Q\overline{j} + R\overline{k}$ is a vector field on R^3 that has continuous second-order partial derivatives, then

$$\nabla \cdot \left(\nabla \times \overline{F}\right) = \operatorname{div} \operatorname{curl} \overline{F} = 0$$

Proof.

$$\nabla \cdot \left(\nabla \times \bar{F} \right) = \nabla \cdot \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \nabla \cdot \left(\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \bar{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \bar{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \bar{k} \right)$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

$$= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial z} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial x} = 0,$$

since mixed partial derivatives do not depend on the order of the variables.