Integrals Over Curves

Assume that C is a (smooth) curve in a plane or space that is parameterized by the vector function $\overline{l}(t)=\langle x(t),y(t),z(t)\rangle$, where $t\in[a,b]$. (Note that if C is a curve in a plane, then we drop the third component $z(t)$.)

Remember that line integrals and path integrals are equivalent.

What does $\int_{\mathrm{C}} dl$ mean?

The integral $\int_{\mathrm{C}} dl$ is a line integral of a scalar function $f(x,y)=1$ or $f(x,y,z)=1$, and represents the length of C.

To compute $\int_{\mathrm{C}} dl$

- In R^2 (a plane),

$$\int_{\mathrm{C}} dl=\int_a^b\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2}\,dt=\int_a^b\left|\overline{l}'(t)\right|dt.$$

- In R^3 (space),

$$\int_{\mathrm{C}} dl=\int_a^b\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2+\left(z'(t)\right)^2}\,dt=\int_a^b\left|\overline{l}'(t)\right|dt.$$

What does $\int_C d\bar{l}$ mean?

The line integral $\int_C d\bar{l}$ is the displacement vector of the curve / path $\bar{l}(t)$. By definition

$$\int_C d\bar{l} = \int_a^b \bar{l}'(t)\,dt,$$

where $\bar{l}'(t)$ is the tangent vector to the trajectory of a particle (or object) represented as a vector function $\bar{l}(t)$.

The total distance along the curve $\bar{l}(t)$ is calculated by $\int_C \left|\bar{l}'(t)\right| dt$.

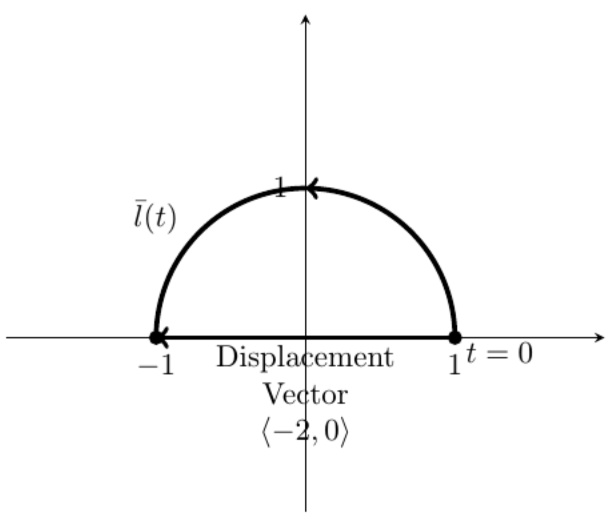
Example. If $\bar{l}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le \pi$, find the displacement of $l(t)$ and the total distance along the curve.

Solution. For the displacement of $l(t)$, we find that $\bar{l}'(t) = \langle -\sin t, \cos t \rangle$ and calculate

$$\int_0^\pi d\bar{l} = \int_0^\pi \bar{l}'(t)\,dt = \int_0^\pi \langle -\sin t,\, \cos t \rangle\, dt = \left\langle \cos t\,|_0^\pi, \sin t\,|_0^\pi \right\rangle = \langle -2, 0 \rangle.$$

For the total distance along the curve, we calculate

$$\int_0^\pi |\,\bar{l}'(t)\,|\,dt = \int_0^\pi \left|\langle -\sin t,\, \cos t \rangle\right| dt = \int_0^\pi \sqrt{(-\sin t)^2 + (\cos t)^2}\,dt = \int_0^\pi 1\,dt = \pi.$$



What does $\int\limits_C f\,ds$ mean?

The integral $\int\limits_C f\,ds$ is the line integral of a continuous real-valued function $f(x,y)$ or $f(x,y,z)$, i.e., it represents the area of the region under the surface $z=f(x,y)$ along the curve $\bar{l}(t)$. We calculate as follows:

$$\int\limits_C f\,ds = \int_a^b f\left(\bar{l}(t)\right)\left|\bar{l}(t)\right|dt.$$

What does $\int dA$ mean where A is a region in the plane.

The integral $\int dA$ is equal to the area of A.

What does $\iint\limits_R f(x,y)\,dA$ mean?

Suppose that $f(x,y)$ is continuous on some region in the xy-plane. The integral $\iint\limits_R f(x,y)\,dA$ represents the volume over the region under the surface which is defined by $z=f(x,y)$. Note that $\iint\limits_R f(x,y)\,dA$ is evaluated as an iterated integral and thus $dA=dxdy$ or $dA=dydx$.

What does $\int \bar{F}\cdot d\bar{A}$ mean?

To calculate $\int \bar{F}\cdot d\bar{A}$, we write the integral as

$$\int \bar{F}\cdot d\bar{A} = \int \bar{F}\cdot \bar{n}\,dA,$$

where $\bar{n}$ is the unit normal of the planar region A and dA is a small patch of the region. Note that $d\bar{A}=\bar{n}dA$.

If $\bar{F}$ is a constant vector field, then we can evaluate the integral as follows.

$$
\begin{aligned}
&\int \bar{F} \cdot d\bar{A} \\
&= \int \bar{F} \cdot \bar{n}\, dA, && d\bar{A} = \bar{n} dA \\
&= \int \left|\bar{F}\right|\left|\bar{n}\right| \cos(\theta)\, dA, && \text{Theorem } \bar{A} \cdot \bar{B} = AB\cos(\theta) \\
&= \int (F)(1)\cos(\theta)\, dA \\
&= F\cos(\theta) \int dA, && F\cos(\theta) \text{is a constant and} \int dA \text{is the area of } A \\
&= FA\cos(\theta)
\end{aligned}
$$

What does $\int_B f(x, y, z)\, dV$ mean where B is a region in space?

In general, there is no interpretation.

If $f(x, y, z) = 1$, then this triple integral represents the volume of B.

If $f(x, y, z)$ represents density, then this triple integral gives the total mass of B.

This triple integral is computed as an iterated integral where $dV = dxdydz$ (or any other order which is convenient to compute).

Line Integrals of Vector Fields (Theory)

Suppose that $\bar{F} = P\bar{i} + Q\bar{j} + R\bar{k}$ is a continuous force field on R^3 (i.e., in 3-dimensions; thus P, Q, and R are functions of x, y, and z). Note that a force field in R^2 (i.e. in 2-dimesions) can be thought of as a special case where $R = 0$ and P and Q depend only on x and y.

Now, a formula can be derived for a force $\bar{F}$ acting upon a particle that moves along the curve C (from some initial point to some terminal point).

The force $\bar{F}$ in moving the particle from P_{i-1} to P_i is approximately

$$\bar{F}\left(x_i^*, y_i^*, z_i^*\right) \cdot \left[\Delta s_i \bar{T}\left(x_i^*, y_i^*, z_i^*\right)\right] = \left[\bar{F}\left(x_i^*, y_i^*, z_i^*\right) \cdot \bar{T}\left(x_i^*, y_i^*, z_i^*\right)\right] \Delta s_i$$

where $\bar{T}\left(x_i^*, y_i^*, z_i^*\right)$ is the unit tangent vector at a point $\left(x_i^*, y_i^*, z_i^*\right)$ between P_{i-1} and P_i on C (details of this will be provided in a vector calculus course; we do not have all concepts available, such as Mean Value Theorem, for a precise derivation). Here, the actual displacement Δs_i along C is approximated by $\Delta s_i \bar{T}\left(x_i^*, y_i^*, z_i^*\right)$.

Moving the particle along C from the initial point to the terminal point is approximately

$$\sum_{i=1}^{n}\left[\bar{F}\left(x_i^*, y_i^*, z_i^*\right)\cdot\bar{T}\left(x_i^*, y_i^*, z_i^*\right)\right]\Delta s_i$$

Recall that the definition of a definite integral is:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\Delta x f\left(x_i^*\right)=\int_a^b f\left(x\right)dx.$$

As n, the number of subintervals, increases, the approximation improves.

Thus, the force field $\bar{F}$ is defined as the limit of the Riemann sums, namely,

$$\int_{\mathrm{C}}\bar{F}\left(x,y,z\right)\cdot\bar{T}\left(x,y,z\right)ds=\int_{\mathrm{C}}\bar{F}\cdot\bar{T}ds$$

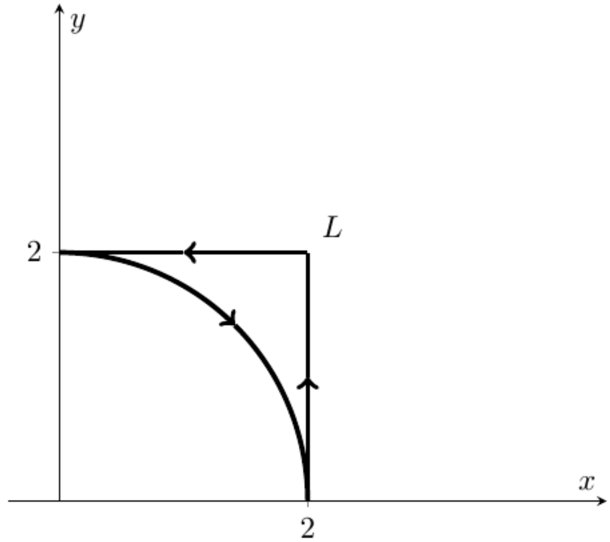
If the curve C is given by the vector equation $\bar{l}\left(t\right)=x\left(t\right)\bar{i}+y\left(t\right)\bar{j}+z\left(t\right)\bar{k}$, where $t\in\left[a,b\right]$, then the tangent unit vector is

$$\bar{T}\left(t\right)=\frac{\bar{l}'\left(t\right)}{\left|\bar{l}'\left(t\right)\right|}.$$

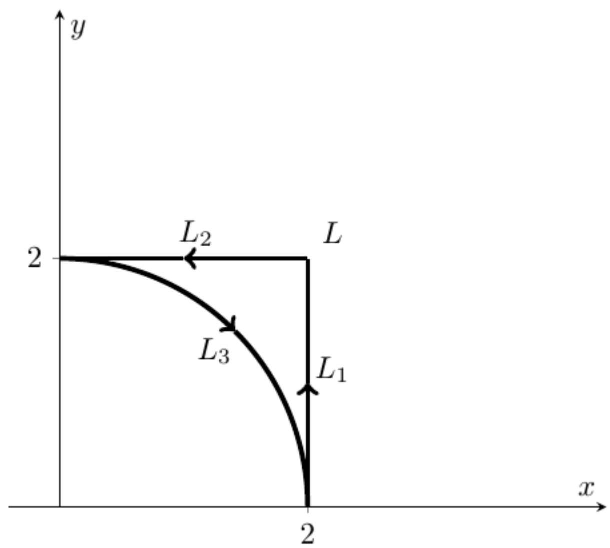
Then, using the definition of a path integral and the fact that $ds=\left|l'\left(t\right)\right|dt$, which follows from the fact that $\frac{ds}{dt}$ is the rate of change of the arc length over time; i.e., the speed of the particle, and since the motion of the particle is described by a position function, we obtain

$$\int_a^b\left[\bar{F}\left(\bar{l}\left(t\right)\right)\cdot\frac{\bar{l}'\left(t\right)}{\left|\bar{l}'\left(t\right)\right|}\right]\left|\bar{l}'\left(t\right)\right|dt=\int_a^b\bar{F}\left(\bar{l}\left(t\right)\right)\cdot\bar{l}'\left(t\right)dt=\int_a^b\bar{F}\cdot d\bar{l}$$

Example. If $\bar{F}=\langle 2xy, y\rangle$, evaluate $\int\bar{F}\cdot d\bar{L}$ around L as shown in the figure below.



Solution. We split L into three pieces as shown and consider each integral separately.



We first consider L_1. We will parametrize this line by $x=2,\ y=t,\ 0\le t\le 2$. Then L_1 is given by $\bar{l}(t)=\langle 2,t\rangle,\ 0\le t\le 2$, and $\bar{l}'(t)=\langle 0,1\rangle$. Evaluating the integral, we obtain

$$\int_{L_1}\bar{F}\cdot d\bar{l}=\int_0^2\bar{F}\left(\bar{l}(t)\right)\cdot\bar{l}'(t)\,dt=\int_0^2\langle 4t,t\rangle\cdot\langle 0,1\rangle\,dt=\int_0^2 t\,dt=\left.\frac{t^2}{2}\right|_0^2=2.$$

Now we consider L_2. We will parameterize this line by $x=2-t,\ y=2,\ 0\le t\le 2$. Then L_2 is given by $\bar{l}(t)=\langle 2-t,2\rangle,\ 0\le t\le 2$, and $\bar{l}'(t)=\langle -1,0\rangle$. Evaluating the integral, we obtain

$$\int_{L_2} \bar{F} \cdot d\bar{l} = \int_0^2 \bar{F}\left(\bar{l}\,(t)\right) \cdot \bar{l}'(t)\,dt = \int_0^2 \langle 4(2-t), 2\rangle \cdot \langle -1, 0\rangle\,dt$$

$$= \int_0^2 -4(2-t)\,dt = -4\int_0^2 (2-t)\,dt = -4\left[2t - \frac{t^2}{2}\right]_0^2 = -8.$$

Finally, we consider L_3. We will parameterize this line by $x = 2\sin t,\ y = 2\cos t,\ 0 \le t \le \frac{\pi}{2}$.

Then L_3 is given by $\bar{l}\,(t) = \langle 2\sin t, 2\cos t\rangle,\ 0 \le t \le \frac{\pi}{2}$, and $\bar{l}'(t) = \langle 2\cos t, -2\sin t\rangle$.

Evaluating the integral, we obtain

$$\int_{L_3} \bar{F} \cdot d\bar{l} = \int_0^{\frac{\pi}{2}} \bar{F}\left(\bar{l}\,(t)\right) \cdot \bar{l}'(t)\,dt$$

$$= \int_0^{\frac{\pi}{2}} \langle 8\sin t\cos t, 2\cos t\rangle \cdot \langle 2\cos t, -2\sin t\rangle\,dt$$

$$= \int_0^{\frac{\pi}{2}} (16\sin t\cos^2 t - 4\sin t\cos t)\,dt.$$

We can evaluate this integral using substitution. Let $u = \cos t$, then $du = -\sin t\,dt$, and we evaluate from $\cos 0 = 1$ to $\cos\frac{\pi}{2} = 0$ and obtain

$$= \int_1^0 \left(-16u^2 + 4u\right)du$$

$$= \left[-\frac{16}{3}u^3 + 2u^2\right]_1^0$$

$$= \frac{16}{3} - 2 = \frac{10}{3}$$

Putting it all together, we obtain

$$\int_L \bar{F} \cdot d\bar{l} = \int_{L_1} \bar{F} \cdot d\bar{l} + \int_{L_2} \bar{F} \cdot d\bar{l} + \int_{L_3} \bar{F} \cdot d\bar{l} = 2 + (-8) + \frac{10}{3} = -\frac{8}{3}.$$