Divergence Theorem

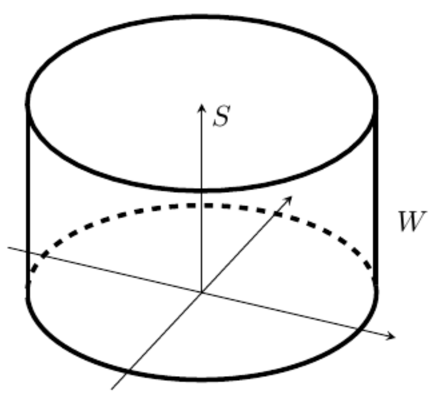
Divergence Theorem

Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\overline{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then,

$$\iint_S \overline{F} \cdot d\overline{S} = \iiint_E \text{div}\overline{F} dV.$$

Thus, the Divergence Theorem states that, under the given conditions, the flux of $\overline{F}$ across the boundary surface of E is equal to the triple integral of the divergence of $\overline{F}$ over E.

Example 1. Let S be the circle given by $x^2 + y^2 \leq 4$ and $z = 5$ and let $\overline{F} = y^2\overline{i} + x^2\overline{j} + z^2\overline{k}$. Compute $\iint_S \overline{F} \cdot d\overline{s}$.



Solution. We use parametric coordinates as follows:

$$\begin{aligned}
&\bar{r}(u,v)=\langle u,v,5\rangle,\\
&\bar{T}_u=\langle 1,0,0\rangle,\\
&\bar{T}_v=\langle 0,1,0\rangle,\\
&\bar{N}=\bar{T}_u\times\bar{T}_v=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & 0 & 0\\ 0 & 1 & 0\end{vmatrix}=\bar{k}.
\end{aligned}$$

We then calculate and obtain

$$\iint_S \bar{F}\cdot d\bar{s}=\iint_{x^2+y^2\le 4}\langle v^2,u^2,25\rangle\cdot\langle 0,0,1\rangle dA=\iint_{x^2+y^2\le 4}25dA=25(16\pi)=400\pi.$$

Example 2. Let W be the cylinder given by $x^2+y^2\le 4$ and $0\le z\le 5$ and let $\bar{F}=xy^2\bar{i}+y^3\bar{j}+4x^2z\bar{k}$. Compute the flux of $\bar{F}$ across the boundary surface of W.

Solution. We begin by calculating the divergence of $\bar{F}$.

$$\text{div}\bar{F}=\frac{\partial xy^2}{\partial x}+\frac{\partial y^3}{\partial y}+\frac{\partial 4x^2z}{\partial z}=y^2+3y^2+4x^2=4y^2+4x^2$$

Then, applying the Divergence Theorem, we obtain

$$\begin{aligned}
\iint_{\text{S}} \bar{F}\cdot d\bar{s} &= \iint_{\text{S}}\langle xy^2,y^3,4x^2z\rangle\cdot d\bar{s}\\
&=\iiint_W \text{div}\bar{F}dV\\
&=\iiint_W (4y^2+4x^2)dV\\
&=4\iint_{x^2+y^2\le 4}\left(\int_0^5 (x^2+y^2)dz\right)dA\\
&=20\iint_{x^2+y^2\le 4}(x^2+y^2)dA\\
&=20\int_0^{2\pi}\int_0^2 r^2 r dr d\theta\\
&=20\int_0^{2\pi}\int_0^2 r^3 dr d\theta\\
&=20\int_0^{2\pi}\left[\frac{1}{4}r^4\right]_0^2 d\theta\\
&=20\times 2\pi\times 4\\
&=160\pi,
\end{aligned}$$

where we've used the polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2, \quad dA = r\,dr\,d\theta.$$