Volumes

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Let S be a three-dimensional solid, placed so that it lies between the vertical planes $x = a$ and $x = b$. If the cross sectional area of S in the plane S_x, through x and perpendicular to the x-axis, is $A(x)$, where

A is a continuous function, then the volume of S is $V = \int_a^b A(x)\,dx$.

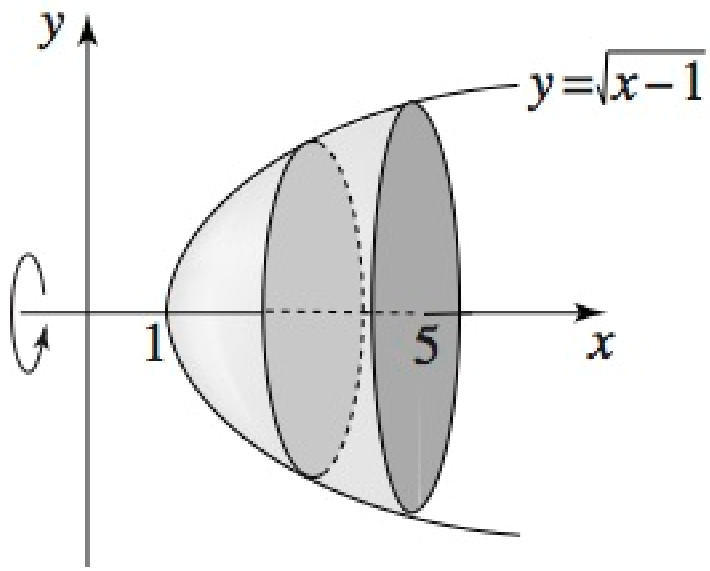
If S is a solid of revolution, then all cross-sections are disks, and $A(x) = \pi r_x^{\,2}$ where r is the radius of the cross-sectional disk in the plane S_x through x.

Rotation about the x-axis or horizontal line

$$V = \int_{x=a}^{x=b} \pi r^2\,dx \quad \text{or} \quad V = \int_{x=a}^{x=b} \pi(r_{outer}^2 - r_{inner}^2)\,dx$$

Example. Find the volume of the solid obtained by rotating the region bounded by the curves $y = \sqrt{x-1}$, $y = 0$, and $x = 5$ about the x-axis.

Solution.



The cross sectional area perpendicular to the x-axis is
$A(x) = \pi(y(x))^2$.

The volume of a slice with thickness dx is
$dV = A(x)dx = \pi(y(x))^2 dx$.

The volume of the given solid of revolution is

$$V = \int_1^5 A(x)\,dx = \int_1^5 \pi y^2\,dx == \pi\int_1^5 (\sqrt{x-1})^2\,dx$$

$$= \pi\int_1^5 (x-1)\,dx = \pi\left(\frac{x^2}{2} - x\right)\Bigg|_1^5$$

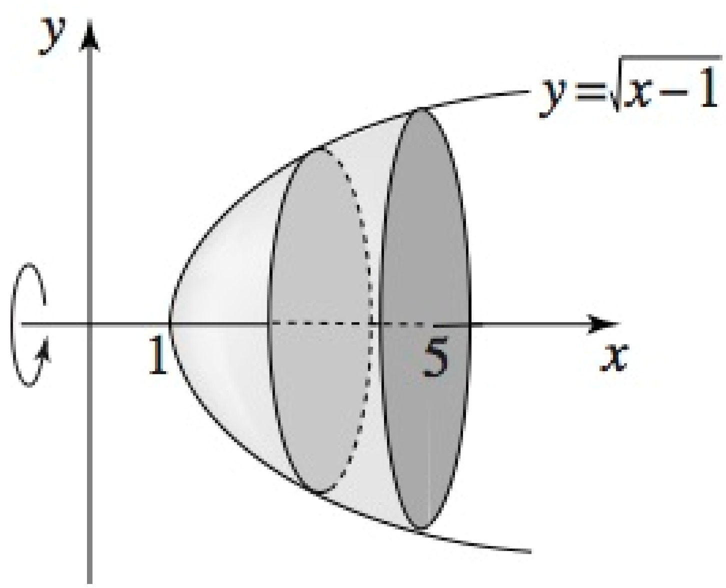
$$= \pi\left[\left(\frac{5^2}{2} - 5\right) - \left(\frac{1^2}{2} - 1\right)\right] = 8\pi$$

Rotation about y-axis or vertical line

$$V = \int_{y=a}^{y=b} \pi r^2\,dy \quad \text{or} \quad V = \int_{y=a}^{y=b} \pi(r_{outer}^2 - r_{inner}^2)\,dy$$

Example. Find the volume of the solid obtained by rotating the region bounded by the curves $x = y^2 + 1$ and $x = 3$ about the line $x = 3$.

Solution.



The cross sectional area perpendicular to the y-axis is

$$A(y) = \pi(3 - x(y))^2$$
$$= \pi(3 - (1 + y^2))^2$$
$$= \pi(2 - y^2)^2$$

The volume of a slice with thickness dy is

$$dV = A(y)dy = \pi(x(y))^2 dy.$$

The volume of the solid object is

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} A(y)\,dy = \int_{-\sqrt{2}}^{\sqrt{2}} \pi(2-y^2)^2\,dy$$

$$= \pi\int_{-\sqrt{2}}^{\sqrt{2}} (4-2y^2+y^4)\,dy$$

$$= \pi\left(4y - \frac{2y^3}{3} + \frac{y^5}{5}\right)\Bigg|_1^5 = \frac{64\sqrt{2}\pi}{15}$$