The Robert Gillespie ACADEMIC SKILLS CENTRE

Series

Series

An expression of the form $a_1 + a_2 + ... + a_n + ... = \sum_{n=1}^{\infty} a_n$ is called an infinite series or simply a series, where a_n is the nth term of a sequence.

The nth partial sum s_n of an infinite series $\sum_{n=1}^{\infty} a_n$ is $s_n = a_1 + a_2 + ... + a_n$.

Definition: An infinite series $\sum_{n=1}^{\infty} a_n$ is convergent if $\lim_{n\to\infty} s_n = s$ for some real number s. The series is divergent if the limit does not exist (i.e., if it is not a real number).

If the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent and $\lim_{n\to\infty} s_n = s$, then s is called the sum of the series and can be written as $s = a_1 + a_2 + ... + a_n + ...$. If the series diverges, then it has no sum.

Geometric Series

The geometric series, $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + ... = \sum_{n=0}^{\infty} ar^n$, converges if $|r| < 1$ and has the sum $\frac{a}{1-r}$ where a is the first term of the series and r is called the common ratio. The geometric series diverges if $|r| \geq 1$.

Example. Determine if the series converges or diverges. If the series converges, find its sum.

(a) $\sum_{n=1}^{\infty} 5^{-n} 2^{n+1}$ (b) $\sum_{n=0}^{\infty} \frac{(-2)^{2n}}{3^{n-1}}$

Solution. (a) Rewrite the series in the form: $\sum_{n=1}^{\infty} ar^{n-1}$

$$\sum_{n=1}^{\infty} 5^{-n} 2^{n+1} = \sum_{n=1}^{\infty} \frac{2^{n+1}}{5^n} = \sum_{n=1}^{\infty} \frac{2^{n+1}}{5^n} \frac{5}{5} \frac{2^2}{2^2}$$

$$= \sum_{n=1}^{\infty} \frac{2^{n-1}}{5^{n-1}} \frac{4}{5} = \sum_{n=1}^{\infty} \frac{4}{5} \left(\frac{2}{5} \right)^{n-1}$$

Then, $a = \frac{4}{5}$ and $r = \frac{2}{5} < 1$. Thus, this geometric series converges, and its sum is $\frac{a}{1-r} = \frac{\frac{4}{5}}{1-\frac{2}{5}} = \frac{4}{3}$.

(b) Rewrite the series in the form: $\sum_{n=0}^{\infty} ar^{n}$

$$\sum_{n=0}^{\infty} \frac{(-2)^{2n}}{3^{n-1}} = \sum_{n=0}^{\infty} 3 \frac{4^n}{3^n} = \sum_{n=0}^{\infty} 3 \left(\frac{4}{3} \right)^{n}$$

Then, $r = \frac{4}{3} > 1$. Thus, this geometric series diverges, and it has no sum.

Telescoping Series

Use partial fraction decomposition for rational expressions to calculate the partial sum; for instance, to find the partial sum of $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$, use $\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$. Then, expand the series up to its nth term to find the partial sum.

Example. Prove that the following infinite series converges, and find its sum:

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\frac{1}{1(2)}+\frac{1}{2(3)}+\frac{1}{3(4)}+\ldots+\frac{1}{n(n+1)}+\ldots$$

Solution. The partial sum of this infinite series is

$$S_n=\sum_{i=1}^{n}\frac{1}{i(i+1)}=\frac{1}{1(2)}+\frac{1}{2(3)}+\frac{1}{3(4)}+\ldots+\frac{1}{n(n+1)}$$

Using partial fraction decomposition on the rational expression of the partial sum: $\frac{1}{i(i+1)}=\frac{1}{i}-\frac{1}{i+1}$

Then, once expanding the sum, the partial sum reduces to n

$$S_n=\sum_{i=1}^{n}\left(\frac{1}{i}-\frac{1}{i+1}\right)=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots$$

$$+\left(\frac{1}{n-2}-\frac{1}{n-1}\right)+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=1-\frac{1}{n+1}$$

Finally, take the limit of the partial sum as n approaches to ∞.

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=\lim_{n\to\infty}\left(1-\frac{1/n}{1+1/n}\right)=1.$$

p-series

Fact: The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series) is divergent since it is a p-series with $p = 1$.