The Robert Gillespie ACADEMIC SKILLS CENTRE

Limits as x approaches Infinity

Limits as $x \to \pm\infty$

When the values of the function $f(x)$ approach the number L as x increases without a bound (meaning that the values of x can be chosen to be larger than any real number chosen),

$$\lim_{x\to+\infty} f(x) = L \ .$$

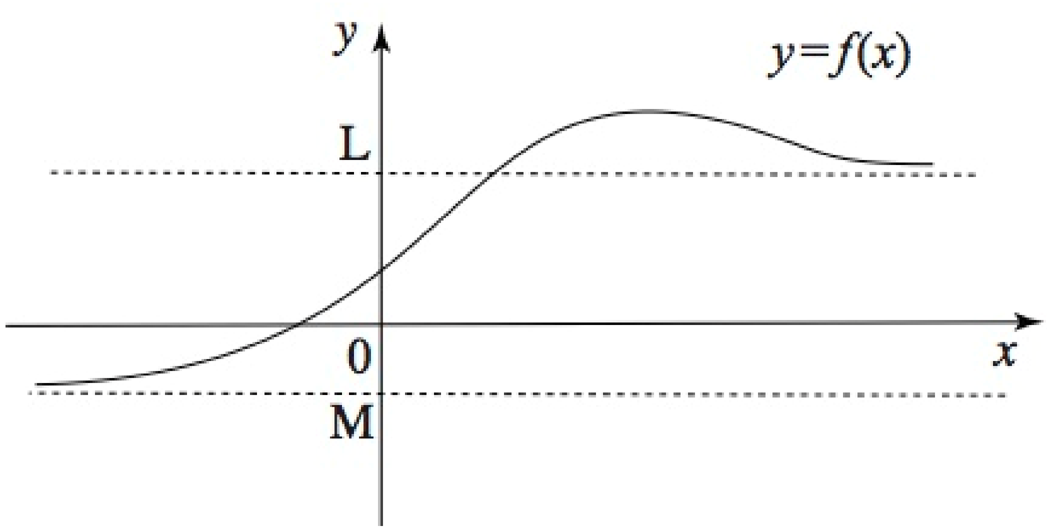
Likewise,

$$\lim_{x\to-\infty} f(x) = M$$

when the functional values $f(x)$ approach the number M as x decreases without a bound.

Horizontal Asymptotes

Geometrically, $\lim_{x\to+\infty} f(x) = L$ means that the graph of $f(x)$ approaches the horizontal line $y = L$ at infinity, while $\lim_{x\to-\infty} f(x) = M$ means that the graph of $f(x)$ approaches the horizontal line $y = M$ at negative infinity.



The lines $y = L$ and $y = M$ are called horizontal asymptotes.

Note that to find horizontal asymptotes, both the limit at $+\infty$ and at $-\infty$ must be checked.

Reciprocal Power Rules

If A and p are constants with $p > 0$ and x^p is defined for all x, then

$$\lim_{x\to\infty}\frac{A}{x^p} = 0 \quad \text{and} \quad \lim_{x\to-\infty}\frac{A}{x^p} = 0\ .$$

Computation of Limits when $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty}\dfrac{p(x)}{q(x)}$

Step 1. Divide each term in $f(x)$ by the highest power of x^p that appears in the denominator. (Alternatively, divide by the highest power in the numerator, or by the highest power overall in $f(x)$.)

Step 2. Compute $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$ using algebraic properties of limits and the reciprocal power rules.

Example. Evaluate $\lim\limits_{x\to\infty}\dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$.

Solution. The highest power in the denominator is x^2.
Divide the numerator and denominator by x^2 to get:

$$\lim_{x\to\infty}\frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty}\frac{3 - 1/x - 2/x^2}{5 + 4/x + 1/x^2} = \frac{3 - 0 - 0}{5 + 0 + 0} = \frac{3}{5}$$

Thus, the graph of $f(x) = \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$ has a horizontal asymptote at $y = \dfrac{3}{5}$. Note that the limit at $-\infty$ gives the same answer.

Example. Find $\lim_{r\to-\infty} \frac{r^4 - r^2 + 1}{r^5 + r^3 - r}$.

Solution. $$\lim_{r\to-\infty} \frac{r^4 - r^2 + 1}{r^5 + r^3 - r} = \lim_{r\to-\infty} \frac{1/r - 1/r^3 + 1/r^5}{1 + 1/r^2 - 1/r^4} = \frac{0 - 0 + 0}{1 + 0 - 0} = 0$$

There is a horizontal asymptote at $y = 0$. Note that the limit at $+\infty$ gives the same answer.

Example. Find $\lim_{x\to+\infty} \frac{x^2 + x}{3 - x}$.

Solution. $$\lim_{x\to+\infty} \frac{x^2 + x}{3 - x} = \lim_{x\to+\infty} \frac{x + 1}{3/x - 1} = -\infty$$

Thus, the graph of $f(x) = \frac{x^2 + x}{3 - x}$ has no horizontal asymptote at $+\infty$.

Note that to find the horizontal asymptotes, the limit at $-\infty$ must be checked as well.