The Robert Gillespie ACADEMIC SKILLS CENTRE

L'Hopital's Rule

L'Hôpital's Rule

Suppose that f and g are differentiable and $g'(x) \neq 0$ on an open interval that contains a (except possibly at a) and

$$\left.\begin{array}{l} \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \\ \text{or} \\ \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty \end{array}\right\} \quad \text{eg. } \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\pm\infty}{\pm\infty},$$

then, $\lim_{x\to a} \frac{f(x)}{g(x)} \overset{\text{H}}{=} \lim_{x\to a} \frac{f'(x)}{g'(x)}$, provided that the limit on the right side exists (or is an indeterminate form, or is positive infinity, or negative infinity).

When applying L'Hôpital's Rule, always try to get the function in the fraction form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

7 Indeterminate Forms

$$\underbrace{\frac{0}{0}, \frac{\infty}{\infty}}_{\text{Case 1}}, \quad \underbrace{\infty - \infty}_{\text{Case 2}}, \quad \underbrace{0 \cdot \infty}_{\text{Case 3}}, \quad \underbrace{1^{\infty}, \; 0^{0}, \; \infty^{0}}_{\text{Case 4}}$$

Case 1. $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$

Example. Evaluate $\lim\limits_{x\to 0}\dfrac{\tan x - x}{x^3}$.

Solution.

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} \overset{\mathrm{H}}{=} \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}$$

$$\overset{\mathrm{H}}{=} \lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x} = \lim_{x\to 0}\frac{\sec^2 x\tan x}{3x}$$

$$\overset{\mathrm{H}}{=} \lim_{x\to 0}\frac{(2\sec x\sec x\tan x)\tan x + (\sec^2 x)\sec x}{3}$$

$$= \frac{0+1}{3} = \frac{1}{3}$$

Case 2. $\infty - \infty$

Rewrite the function as one fraction (for instance, compute the common denominator) so that it is in the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, and then apply L'Hôpital's Rule.

Example. Evaluate $\lim\limits_{x\to 1^+}\left(\dfrac{1}{\ln x} - \dfrac{1}{x-1}\right)$.

Solution.

$$\lim_{x\to 1^+} \frac{1}{\ln x} - \frac{1}{x-1} = \lim_{x\to 1^+} \frac{x-1-\ln x}{(x-1)\ln x} \overset{\mathrm{H}}{=} \lim_{x\to 1^+} \frac{1-\frac{1}{x}}{\ln x + (x-1)\frac{1}{x}}$$

$$= \lim_{x\to 1^+} \frac{\frac{x-1}{x}}{\frac{x\ln x + x - 1}{x}} = \lim_{x\to 1^+} \frac{x-1}{x\ln x + x - 1}$$

$$\overset{\mathrm{H}}{=} \lim_{x\to 1^+} \frac{1}{\ln x + 1 + 1} = \frac{1}{2}$$

<u>Case 3.</u> $\infty \cdot 0$

Rewrite the function by taking the reciprocal of one of the terms (recall, from algebra, $AB = \frac{A}{1/B} = \frac{B}{1/A}$); thus, not changing the function, so that it is in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and then apply L'Hôpital's Rule.

Example. Evaluate $\lim_{x\to 0^+} x^2 \ln x$.

Solution.

$$\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{\ln x}{x^{-2}} \overset{\mathrm{H}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-2x^{-3}}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim_{x\to 0^+} \left(\frac{1}{x}\right)\left(\frac{x^3}{-2}\right)$$

$$= \lim_{x\to 0^+} \frac{-x^2}{2} = 0$$

<u>Case 4.</u> 1^∞, 0^0, or ∞^0

Step 1. Take the natural logarithm of the function and use the logarithmic rule $\ln(A^B) = B\ln(A)$.

Bring the resulting function into the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, and then apply L'Hôpital's Rule.

Step 2. Apply the inverse property $e^{\ln(x)} = x$ on the result of Step 1.

Example. Evaluate $\lim\limits_{x\to\infty}\left(1+\dfrac{5}{x^2}\right)^x$.

Solution. Step 1.

$$y = \lim_{x\to\infty}\left(1+\frac{5}{x^2}\right)^x$$

Note that in the first step below the limit and the logarithmic function are exchanged.

$$\ln y = \lim_{x\to\infty}\ln\left(1+\frac{5}{x^2}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{5}{x^2}\right)$$

$$= \lim_{x\to\infty}\frac{\ln\left(1+\frac{5}{x^2}\right)}{\frac{1}{x}} \overset{\mathrm{H}}{=} \lim_{x\to\infty}\frac{\frac{1}{\left(1+\frac{5}{x^2}\right)}\left(-10x^{-3}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{10}{x\left(1+\frac{5}{x^2}\right)} = \lim_{x\to\infty}\frac{10}{x+\frac{5}{x}} = \lim_{x\to\infty}\frac{10}{\frac{x^2+5}{x}}$$

$$= \lim_{x\to\infty}\frac{10x}{x^2+5} \overset{\mathrm{H}}{=} \lim_{x\to\infty}\frac{10}{2x} = 0$$

Step 2.

$$\ln y = 0$$
$$e^{\ln y} = e^0$$
$$y = 1$$

Thus, $\lim_{x\to\infty}\left(1+\frac{5}{x^2}\right)^x = 1.$