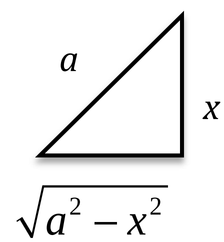
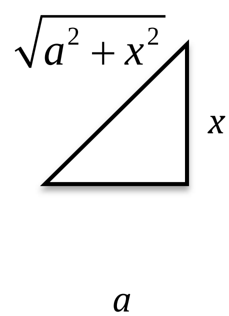
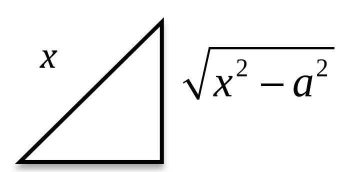
# Integration by Trigonometric Substitution

## Trigonometric Substitution

| Expression | Substitution Needed | Identity Needed | Right Angle Triangle |
|---|---|---|---|
| $\sqrt{a^2 - x^2}$ | $x = a\sin\theta$<br>$dx = a\cos\theta d\theta$ | $1 - \sin^2\theta = \cos^2\theta$ |  $\sin\theta = \frac{opp}{hyp} = \frac{x}{a}$ |
| $\sqrt{a^2 + x^2}$ | $x = a\tan\theta$<br>$dx = a\sec^2\theta d\theta$ | $1 + \tan^2\theta = \sec^2\theta$ |  $\tan\theta = \frac{opp}{adj} = \frac{x}{a}$ |
| $\sqrt{x^2 - a^2}$ | $x = a\sec\theta$<br>$dx = a\sec\theta\tan\theta d\theta$ | $\sec^2\theta - 1 = \tan^2\theta$ |  |

$a$

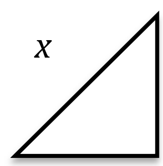
$$\sec\theta = \frac{hyp}{adj} = \frac{x}{a}$$

Example. Compute $\int \frac{\sqrt{x^2-9}}{x}dx$.

Solution. Let $x = 3\sec\theta$. Then, $dx = 3\sec\theta\tan\theta d\theta$

The trigonometric identity needed is $\sec^2\theta - 1 = \tan^2\theta$.

The right angle triangle needed is:




$$\begin{aligned}
\int \frac{\sqrt{x^2-9}}{x}dx &= \int \frac{\sqrt{(3\sec\theta)^2-9}}{3\sec\theta}3\sec\theta\tan\theta d\theta \\
&= \int \sqrt{9\sec^2\theta - 9}\,\tan\theta d\theta \\
&= \int \sqrt{9\tan^2\theta}\,\tan\theta d\theta \\
&= \int 3\tan\theta\tan\theta\, d\theta = 3\int \tan^2\theta\, d\theta \\
&= 3\int (\sec^2\theta - 1)\, d\theta = 3(\tan\theta - \theta) + C \\
&= 3\left(\frac{\sqrt{x^2-9}}{3} - \operatorname{arc}\sec(x/3)\right) + C
\end{aligned}$$

Note that $\theta = \text{arcsec}(x/3)$ (from the substitution equation).