The Robert Gillespie ACADEMIC SKILLS CENTRE

Integration by Substitution

Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and if f is continuous on I, then $\int f(g(x))g'(x)\,dx = \int f(u)\,du$.

Note that the assumptions guarantee that the integrands on both sides of this equality are continuous functions.

TIP: Substitute terms that are:

- raised to a high power
  $\int (3x^2 + 2x)(x^3 + x^2)^3\,dx$ let $u = x^3 + x^2$
- under a root
  $\int 4x^3 \sqrt[5]{x^4 - 1}\,dx$ let $u = x^4 - 1$
- in the power of the exponential
  $\int e^{x^3 + x}(3x^2 + 1)\,dx$ let $u = x^3 + x$
- within the trigonometric function
  $\int 3x^2 \sin(x^3)\,dx$ let $u = x^3$
- within the logarithmic function
  $\int -3(4x^3 - x^2)\ln(-3x^4 + x^3)\,dx$ let $t = -3x^4 + x^3$

Example. Integrate $\int 3(8z + 1)e^{4z^2 + z}\,dz$.

Solution. Let $u = 4z^2 + z$. Then, $du = (8z + 1)dz$.

$$\int 3(8z+1)e^{4z^2+z}\,dz = \int 3e^u\,du = 3\int e^u\,du$$
$$= 3e^u + C = 3e^{4z^2+z} + C$$

Example. Integrate $\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

Solution. Let $u = 1-4x^2$. Then, $du = -8xdx$, but rewrite it as
$$-\frac{1}{8}du = xdx$$

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = \int x(1-4x^2)^{-\frac{1}{2}}\,dx = -\frac{1}{8}\int u^{-\frac{1}{2}}\,du$$
$$= -\frac{1}{4}u^{\frac{1}{2}} + C = -\frac{1}{4}(1-4x^2)^{\frac{1}{2}} + C$$

Definite Integral by Substitution

If $g'(x)$ is continuous on $[a,b]$ and if f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$
$$= F(u)\Big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a)),$$

where F is an antiderivative of f. (Note that an alternative solution is provided to the example below for illustration of this formula.)

Computing a Definite Integral by Substitution

Step 1. Solve the integral as an indefinite integral.

Step 2. Use the result of the indefinite integral, and evaluate it over the interval of integration.

Alternatively, change the limits of integration when converting the integral to the new variable u (see example below).

Example. Integrate $\int_{-1}^{0} \frac{2x}{(2+x^2)^3}\,dx$.

Solution. Step 1.

Let $u = 2 + x^2$. Then, $du = 2x\,dx$.

$$\int \frac{2x}{(2+x^2)^3}\,dx = \int 2x(2+x^2)^{-3}\,dx = \int u^{-3}\,du$$

$$= -\frac{1}{2}u^{-2} + C = -\frac{1}{2}(2+x^2)^{-2} + C$$

Step 2.

$$\int_{-1}^{0} \frac{2x}{(2+x^2)^3}\,dx = -\frac{1}{2}(2+x^2)^{-2}\Big|_{-1}^{0}$$

$$= \left[-\frac{1}{2}(2+(0)^2)^{-2}\right] - \left[-\frac{1}{2}(2+(-1)^2)^{-2}\right]$$

$$= \frac{1}{2}\left[-(2)^{-2} + (3)^{-2}\right] = \frac{1}{2}\left[-\frac{1}{4} + \frac{1}{9}\right] = -\frac{5}{72}$$

Alternative solution.

Let $u = 2 + x^2$. Then, $du = 2x\,dx$.

$$\int_{-1}^{0} \frac{2x}{(2+x^2)^3}\,dx = \int_{-1}^{0} 2x(2+x^2)^{-3}\,dx = \int_{3}^{2} u^{-3}\,du$$

$$= -\frac{1}{2}u^{-2}\Big|_{3}^{2} = -\frac{1}{2}\left[\frac{1}{4} - \frac{1}{9}\right] = -\frac{5}{72}$$