Integral Test

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Suppose f is continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$.

Then, the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent.

Remember: By the Integral Test, the following result can be proven.

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$. For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series) is divergent since it is a p-series with $p = 1$.

Example. Determine whether the series $\sum_{n=1}^{\infty} ne^{-n^2}$ diverges or converges.

Solution. Let $f(x) = xe^{-x^2}$. The function $f(x)$ is continuous, positive, and decreasing, since $f'(x) = e^{-x^2}(1-2x) < 0$ for all x values in $[1,\infty)$.

By the integral test, (using integration by substitution)

$$\int_1^{\infty} xe^{-x^2}\,dx = \lim_{t\to\infty}\int_1^{t} xe^{-x^2}\,dx = \lim_{t\to\infty} -\frac{e^{-x^2}}{2}\Bigg|_1^t$$

$$= \lim_{t\to\infty}\left(-\frac{e^{-t^2}}{2}\right) - \left(-\frac{e^{-1}}{2}\right) = \frac{e^{-1}}{2} = \frac{1}{2e}$$

since $\lim_{x \to \infty} e^{-x} = 0$. Thus, the series $\sum_{n=1}^{\infty} ne^{-n^2}$ converges.