The Robert Gillespie ACADEMIC SKILLS CENTRE

Improper Integrals

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Type I. Infinite Intervals

(a) If f is continuous on $[a,\infty)$, then $\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$.

(b) If f is continuous on $(-\infty,a]$, then $\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx$.

(c) If f is continuous on $(-\infty,\infty)$, then break up the integral at a convenient value of a, i.e., write $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$, and then use (a) and (b).

Type II. Discontinuous Functions

(a) If f is continuous on $[a,b)$ and discontinuous at b, then $\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$.

(b) If f is continuous on $(a,b]$ and discontinuous at a, then $\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$.

(c) If f is continuous on $[a,b]$, except at a number c in (a,b), then break up the integral at c, i.e., write $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, and then use (a) and (b).

The improper integral is said to:

- **CONVERGE** if the limit in (a) and (b) exists, or if both limits in (c) exist
- **DIVERGE** if the limit in (a) or (b) does not exist, or if at least one of the limits in (c) does not exist

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Remember: The improper integral $\int_1^{\infty} \frac{1}{x^p}\,dx$, where p is a real number converges when $p > 1$, and diverges when $p \le 1$.

Example. Determine whether the following integrals converge or diverge.

(a) $\int_2^{\infty} e^{3x} dx$ (b) $\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx$

(c) $\int_0^1 x \ln x \, dx$

Solution. (a)

$$\int_2^{\infty} e^{3x}\,dx = \lim_{t\to\infty} \int_2^t e^{3x}\,dx = \lim_{t\to\infty} \left(\frac{e^{3x}}{3}\right)\Bigg|_2^t$$

$$= \lim_{t\to\infty} \left[\left(\frac{e^{3t}}{3}\right) - \left(\frac{e^6}{3}\right)\right] = +\infty$$

Thus, $\int_2^{\infty} e^{3x}\,dx$ diverges.

(b)

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = \int_{-\infty}^{0} \frac{1}{x^2+1} dx + \int_0^{\infty} \frac{1}{x^2+1} dx$$

$$= \lim_{t\to-\infty} \int_t^0 \frac{1}{x^2+1} dx + \lim_{s\to\infty} \int_0^s \frac{1}{x^2+1} dx$$

$$= \lim_{t\to-\infty} \left(\arctan(x)\right)\Big|_t^0 + \lim_{s\to\infty} \left(\arctan(x)\Big|\right)_0^s$$

$$= \lim_{t\to-\infty} \left(\arctan(0) - \arctan(t)\right) + \lim_{s\to\infty} \left(\arctan(s) - \arctan(0)\right)$$

$$= \left(0 + \frac{\pi}{2}\right) + \left(\frac{\pi}{2} - 0\right) = \pi$$

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Thus, $\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx$ converges.

Note that $\arctan(x)$ tends to $\pi/2$ as x approaches $+\infty$ and to $-\pi/2$ as x approaches $-\infty$.

(c) Compute the indefinite integral using integration by parts where

$u = \ln x$ and $dv = x dx$.

$$\int x \ln x \, dx = \frac{x^2 \ln x}{2} - \int \left(\frac{x^2}{2}\right)\left(\frac{1}{x} dx\right)$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2}\int x \, dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$$

Now, solve the improper integral.

$$\int_0^1 x \ln x \, dx = \lim_{t \to 0^+} \int_t^1 x \ln x \, dx$$

$$= \lim_{t \to 0^+} \left(\frac{x^2 \ln x}{2} - \frac{x^2}{4}\right)\Bigg|_t^1$$

$$= \lim_{t \to 0^+} \left(\frac{(1)^2 \ln(1)}{2} - \frac{(1)^2}{4}\right) - \left(\frac{t^2 \ln t}{2} - \frac{t^2}{4}\right)$$

$$= \left(0 - \frac{1}{4}\right) - (0 - 0) = -\frac{1}{4}$$

Thus, $\int_0^1 x \ln x \, dx$ converges.

Note that, applying L'Hôpital's Rule to solve the limit,

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$$\lim_{t\to 0^+} t^2 \ln t = \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t^2}} = \lim_{t\to 0^+} \frac{\ln t}{t^{-2}} \overset{H}{=} \lim_{t\to 0^+} \frac{\frac{1}{t}}{-2t^{-3}} = \lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{-2}{t^3}}$$

$$= \lim_{t\to 0^+} \left(\frac{1}{t}\right)\left(\frac{t^3}{-2}\right) = \lim_{t\to 0^+} \left(-\frac{t^2}{2}\right) = 0$$

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