How to Graph a Function

A General Procedure for Sketching the Graph of a Function

Step 1. Find the domain of $y = f(x)$ (that is, all x values where $f(x)$ is defined).

Step 2. Find and plot all intercepts. The y-intercept (where $x = 0$) is usually easy to find, but the x-intercept (where $y = f(x) = 0$) may be difficult to find as it involves solving an equation.

Step 3. Determine all vertical and horizontal asymptotes of the graph of $f(x)$. Draw the asymptotes by using dashed lines.

def: Vertical Asymptote
The line $x = c$ is a vertical asymptote of the graph of $f(x)$ if either

$$\lim_{x \to c^-} f(x) = +\infty \quad (\text{or } -\infty)$$

or $$\lim_{x \to c^+} f(x) = +\infty \quad (\text{or } -\infty)$$

def: Horizontal Asymptote
The horizontal line $y = b$ is called a horizontal asymptote of the graph of $y = f(x)$ if

$$\lim_{x \to -\infty} f(x) = b \quad \text{or} \quad \lim_{x \to \infty} f(x) = b.$$

Step 4. Find $f'(x)$ and use it to determine the critical numbers of $f(x)$ and the intervals where $f(x)$ is increasing and decreasing.

Procedure for Using the Derivative to Determine Intervals of Increase and Decrease for a Function f

Step I. Find the critical numbers, i.e., all values of x for which $f'(x) = 0$ or $f'(x)$ does not exist, and mark these numbers on a number line. As well, mark the numbers which are not in the domain of $f(x)$. All these numbers divide the number line into open intervals.

Step II. Choose a test number c from each interval $a < x < b$ determined in Step I., and evaluate $f'(c)$. Then,

If $f'(c) > 0$, the function $f(x)$ is increasing (graph rising) on $a < x < b$.

If $f'(c) < 0$, the function $f(x)$ is decreasing (graph falling) on $a < x < b$.

Step 5. Determine the x and y coordinates of all relative extrema.

def: The First Derivative Test for Relative Extrema
Let c be a critical number for $f(x)$ (that is, $f(c)$ is defined and either or $f'(c)$ does not exist). Then the critical point $P(c, f(c))$ is

a relative maximum - if $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c

a relative minimum - if $f'(x) < 0$ to the left of c and $f'(x) > 0$ to the right of c

not a relative extremum - if $f'(x)$ has the same sign on both sides of c

Step 6. Find $f''(x)$ and use it to determine intervals of concavity and points of inflection.

<u>Second Derivative Procedure for Determining Intervals of Concavity for a Function</u> f

Step I. Find all values of x for which $f''(x) = 0$ or $f''(x)$ does not exist, and mark these numbers on a number line. As well, mark the numbers which are not in the domain of $f(x)$. All these numbers divide the number line into open intervals.

Step II. Choose a test number c from each interval $a < x < b$ determined in Step I., and evaluate $f''(c)$. Then,

If $f''(c) > 0$, the graph of $f(x)$ is concave upward on $a < x < b$.

If $f''(c) < 0$, the graph of $f(x)$ is concave downward on $a < x < b$.

Procedure for Finding the Inflection Points of a Function f

Step I. Compute $f''(x)$ and determine all points in the domain of f where either $f''(c) = 0$ or $f''(c)$ does not exist.

Step II. For each number c found in Step I., determine the sign of $f''(x)$ to the left and to the right of $x = c$, that is, for $x < c$, and for $x > c$.

If $f''(x) > 0$ on one side of $x = c$ and $f''(x) < 0$ on the other side, then $(c, f(c))$ is an inflection point of f.

Step 7. Sketch by putting together all the information gathered from Steps 1-6. Be sure to remember that the graph cannot cross a vertical asymptote, but it can cross its horizontal asymptote, just not at negative infinity and positive infinity.

Example. Sketch the curve $y = \dfrac{1}{x^3 - x}$.

Solution. 1. For $y = \dfrac{1}{x^3 - x} = \dfrac{1}{x(x^2 - 1)}$, the domain consists of all x values except $x = -1$, $x = 0$, and $x = 1$.

2. There are no x-intercepts and y-intercepts.

3. $$\lim_{x \to \pm\infty} \frac{1}{x^3 - x} = \lim_{x \to \pm\infty} \frac{1/x^3}{1 - 1/x^2} = 0$$

Thus, the line $y = 0$ is a horizontal asymptote.

Since the denominator equals 0 when $x=-1$, $x=0$, and $x=1$, and

$$\lim_{x\to -1^+}\frac{1}{x^3-x}=+\infty \qquad \lim_{x\to -1^-}\frac{1}{x^3-x}=-\infty$$

$$\lim_{x\to 0^+}\frac{1}{x^3-x}=-\infty \qquad \lim_{x\to 0^-}\frac{1}{x^3-x}=+\infty$$

$$\lim_{x\to 1^+}\frac{1}{x^3-x}=+\infty \qquad \lim_{x\to 1^-}\frac{1}{x^3-x}=-\infty$$

The lines $x=-1$, $x=0$, and $x=1$ are vertical asymptotes.

4. $y'=\dfrac{-3x^2+1}{(x^3-x)^2}$. Set $y'=0$. The solutions (i.e., the critical numbers) are $x=\pm\dfrac{1}{\sqrt{3}}$.

Note that y' is not defined at $x=-1$, $x=0$, and $x=1$, but these are not critical numbers as they are not in the domain of $f(x)$. However, these x values must be included in the analysis of intervals where the function is increasing and decreasing.

Note that $x=1/\sqrt{3}\approx 0.58$ and $x=-1/\sqrt{3}\approx -0.58$.

	-1		-0.58		0		0.58		1	
dec		dec		inc		inc		dec		dec
$f' < 0$		$f' < 0$		$f' > 0$		$f' > 0$		$f' < 0$		$f' < 0$

The curve is increasing (inc) on $(\frac{-1}{\sqrt{3}},0)$ and $(0,\frac{1}{\sqrt{3}})$. The curve is decreasing (dec) on $(-\infty,-1)$, $(-1,\frac{-1}{\sqrt{3}})$, $(\frac{1}{\sqrt{3}},1)$, and $(1,\infty)$.

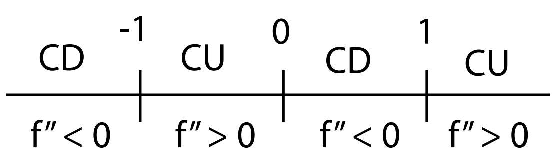
5. The critical numbers are $x = \pm\frac{1}{\sqrt{3}}$.

The point $\left(\frac{1}{\sqrt{3}}, -\frac{3\sqrt{3}}{2}\right)$ is a relative maximum.

The point $\left(-\frac{1}{\sqrt{3}}, \frac{3\sqrt{3}}{2}\right)$ is a relative minimum.

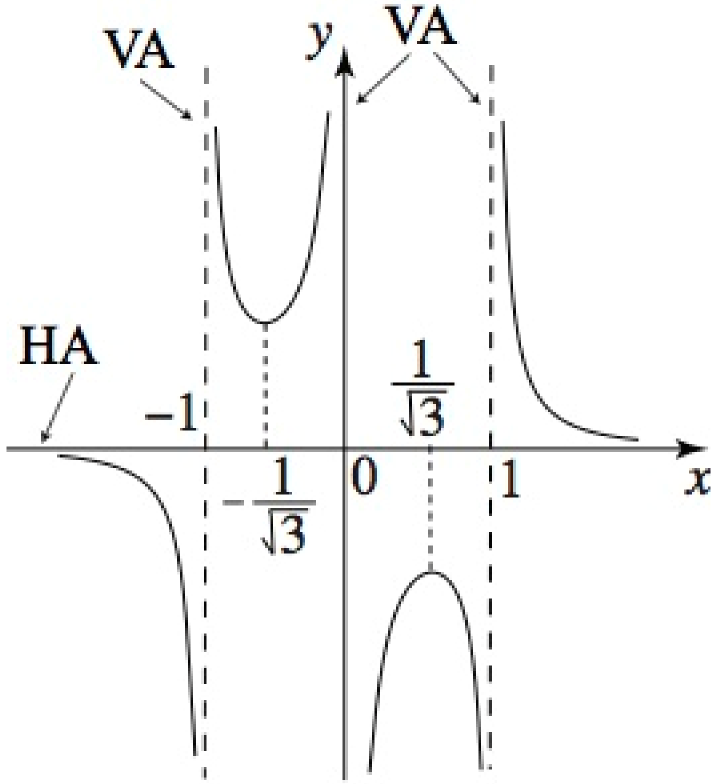
6. $y'' = \dfrac{2[6x^4 + 9x^2 + 1]}{(x^3 - x)^3}$

Set $y'' = 0$. There are no solutions (thus, there are no inflection points). However, for the purpose of analyzing concavity, the numbers $x = -1$, $x = 0$, and $x = 1$ are considered which are not in the domain of $f(x)$.



The curve is concave up (CU) on $(-1, 0)$ and $(1, \infty)$ and is concave down (CD) on $(-\infty, -1)$ and $(0, 1)$.

7. Sketch of $f(x)$.



VA
y
VA
HA
−1
$\frac{1}{\sqrt{3}}$
$-\frac{1}{\sqrt{3}}$
0
1
x