Divergence Test

Divergence Test

If $\lim\limits_{n\to\infty} a_n \neq 0$ or does not exist, then the infinite series $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

If $\lim\limits_{n\to\infty} a_n = 0$, then nothing can be said about the convergence of the infinite series $\sum\limits_{n=1}^{\infty} a_n$, i.e., the series could converge or diverge.

Example. Determine whether each infinite series converges or diverges.

(a) $\sum\limits_{n=1}^{\infty} \arctan(n)$ (b) $\sum\limits_{n=1}^{\infty} e^n$

(c) $\sum\limits_{n=1}^{\infty} e^{-n}$ (d) $\sum\limits_{n=1}^{\infty} \frac{n+1}{n}$

Solution. (a) Let $a_n = \arctan(n)$.

$$\lim_{x\to\infty} a_n = \lim_{x\to\infty} \arctan(n) = \frac{\pi}{2} \neq 0$$

By the Divergence Test, the series diverges.

(b) Let $a_n = e^n$.

$$\lim_{x\to\infty} a_n = \lim_{x\to\infty} e^n = \infty \neq 0$$

By the Divergence Test, the series diverges.

(c) Let $a_n = e^{-n}$.

$$\lim_{x\to\infty} a_n = \lim_{x\to\infty} e^{-n} = 0$$

By the Divergence Test, no conclusion can be made about the convergence of the series. Try another series test.

(d) Let $a_n = \dfrac{n+1}{n}$.

$$\lim_{x\to\infty} a_n = \lim_{x\to\infty} \frac{n+1}{n} = \lim_{x\to\infty} \frac{1+1/n}{1} = 1 \neq 0$$

By the Divergence Test, the series diverges.