The Robert Gillespie ACADEMIC SKILLS CENTRE

Comparison Test & Limit Comparison Test

Comparison Test

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

(i) If $\sum_{n=1}^{\infty} b_n$ converges and $a_n \leq b_n$ for every integer $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\sum_{n=1}^{\infty} b_n$ diverges and $a_n \geq b_n$ for every integer $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example. Determine whether each series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+5^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{\sqrt{n^3}+1}$

Solution. (a) To find b_n, focus on the denominator of a_n to construct an inequality that is true for all $n \geq 1$.

$$2+5^n > 5^n$$

$$\frac{1}{5^n} > \frac{1}{2+5^n}$$

Take $b_n = \dfrac{1}{5^n}$. The series $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{5^n}$ which is a geometric series with $r = \dfrac{1}{5} < 1$, and thus, converges.

By the Comparison Test Part (i), the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+5^n}$ converges as well.

(b) To find b_n, focus on both the numerator and denominator of a_n to construct an inequality that is true for all $n \geq 1$.

$$\frac{3n}{\sqrt{n^3+1}} > \frac{n}{n^{3/2}} = \frac{1}{n^{1/2}}$$

Take $b_n = \frac{1}{n^{1/2}}$. The series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ which is a p-series with $p = \frac{1}{2} < 1$, and thus, diverges.

By the Comparison Test Part (ii), the series $\sum_{n=1}^{\infty} \frac{3n}{\sqrt{n^3+1}}$ diverges.

Limit Comparison Test

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms and $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, where c is a real number, then either both series converge or both diverge.

Note that the Limit Comparison Test is, in many cases, more convenient to use than the Comparison Test, as it does not require to create an inequality.

Example. Determine whether each series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{1}{4n^2+1}$ (b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}}$

Solution. (a) Let $a_n = \frac{1}{4n^2+1}$ and $b_n = \frac{1}{n^2}$.

Then,

$$\lim_{x\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\dfrac{1}{4n^2+1}}{\dfrac{1}{n^2}}=\lim_{n\to\infty}\frac{n^2}{4n^2+1}$$

$$=\lim_{n\to\infty}\frac{1}{4+1/n^2}=\frac{1}{4}>0$$

In this case, $c=\frac{1}{4}>0$. Since $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a p-series with $p=2>1$, it converges, and therefore, the series $\sum_{n=1}^{\infty}\frac{1}{4n^2+1}$ converges by the Limit Comparison Test.

(b) Let $a_n=\frac{1}{\sqrt[3]{n^2+1}}$ and $b_n=\frac{1}{n^{2/3}}$.

Then,

$$\lim_{x\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\dfrac{1}{\sqrt[3]{n^2+1}}}{\dfrac{1}{n^{2/3}}}=\lim_{n\to\infty}\frac{n^{2/3}}{\sqrt[3]{n^2+1}}$$

$$=\lim_{n\to\infty}\frac{n^{2/3}}{\sqrt[3]{n^2(1+1/n^2)}}=\lim_{n\to\infty}\frac{n^{2/3}}{\sqrt[3]{n^2}\,\sqrt[3]{1+1/n^2}}$$

$$=\lim_{n\to\infty}\frac{1}{\sqrt[3]{1+1/n^2}}=1>0$$

In this case, $c=1>0$. Since $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n^{2/3}}$ is a p-series with $p=\frac{2}{3}<1$, it diverges, and therefore, the series $\sum_{n=1}^{\infty}\frac{1}{4n^2+1}$ diverges by the Limit Comparison Test.