Area between Curves

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The area of the region between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is $A = \int_a^b |f(x) - g(x)| dx$.

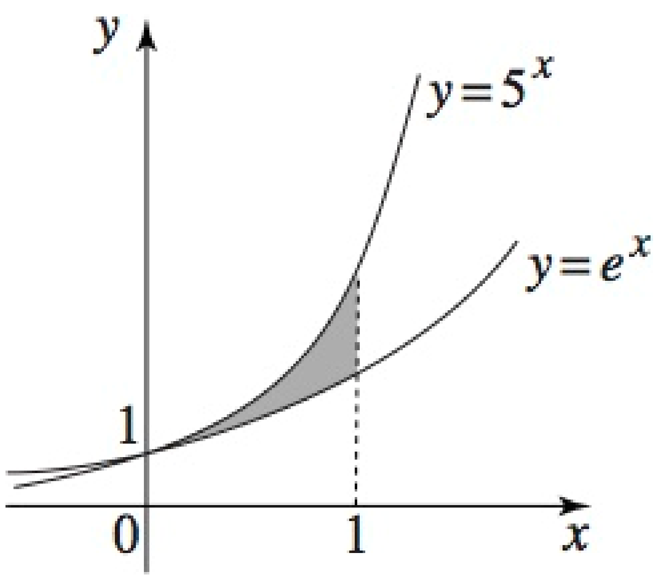
Keep in mind that the term "region between the curves" refers to a bounded region (i.e., to a region that does not extend to infinity).

Instead of the absolute value, if the graphs of the two functions bounding the region do not intersect within (a, b), then

$$A = \int_{x=a}^{x=b} y(x)_{upper} - y(x)_{lower}\, dx$$

Example. Find the area of the region bounded by the curves $y = e^x$, $y = 5^x$, and $x = 1$.

Solution. Sketch the curves and identify the region. Note that the curves intersect at $x = 0$.



On $[0,1]$, the upper curve is $y = 5^x$, and the lower curve is $y = e^x$.

The area of the region is.

$$A = \int_0^1 \left(5^x - e^x\right) dx = \left(\frac{5^x}{\ln 5} - e^x\right)\Bigg|_0^1$$

$$= \left(\frac{5^1}{\ln 5} - e^1\right) - \left(\frac{5^0}{\ln 5} - e^0\right) = \frac{5}{\ln 5} - e - \frac{1}{\ln 5} + 1$$

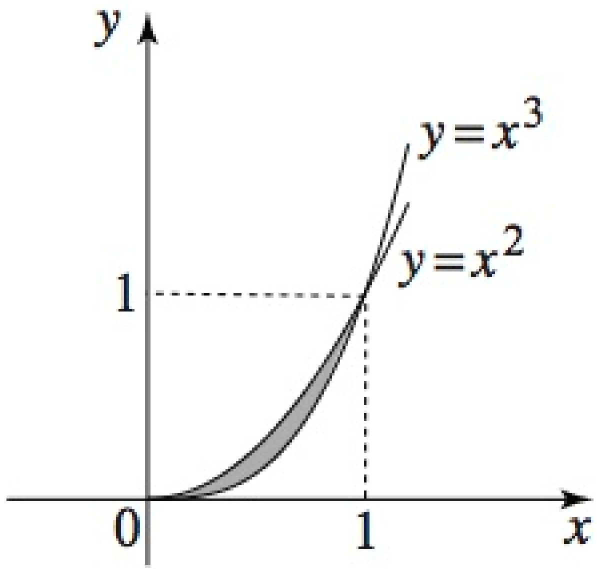
$$= \frac{4}{\ln 5} - e + 1$$

If the region involved looks simpler when the bounding curves are viewed as functions of y, then the analogous formula is used to integrate with respect to y:

$$A = \int_{y=a}^{y=b} x(y)_{right} - x(y)_{left}\, dy$$

Example. Find the area of the region enclosed by the curves $y = x^3$ and $y = x^2$.

Solution. Sketch the area of the region.



To find the points where the curves intersect, combine the two equations:

$$x^3 = x^2$$
$$x^3 - x^2 = 0$$
$$x^2(x-1) = 0$$

Thus, $x = 0$ and $x = 1$. The corresponding points are $(0,0)$ and $(1,1)$.

The right curve is $y = x^3$, or $x = y^{1/3}$, and the right curve is $y = x^2$, or $x = \sqrt{y}$.

The area of the region is

$$A = \int_0^1 (y^{1/3} - y^{1/2})\,dy = \left(\frac{3}{4}y^{4/3} - \frac{2}{3}y^{3/2}\right)\Bigg|_0^1$$
$$= \left(\frac{3}{4}(1)^{4/3} - \frac{2}{3}(1)^{3/2}\right) - \left(\frac{3}{4}(0)^{4/3} - \frac{2}{3}(0)^{3/2}\right) = \frac{1}{12}$$