The Robert Gillespie ACADEMIC SKILLS CENTRE

Alternating Series Test

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If the terms of the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n = b_1 - b_2 + b_3 - \ldots$ where $b_n > 0$ satisfy

(1) $b_{n+1} \le b_n$ for all $n \ge 1$ (b_n is decreasing)

(2) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Example. Determine whether the following series are convergent or divergent.

(a) $\sum_{n=2}^{\infty}\frac{n(-1)^n}{\ln n}$ (b) $\sum_{n=2}^{\infty}\frac{\cos(n\pi)}{n^{3/4}}$

Solution. (a) Checking condition (2) of the alternating series test.

Let $b_n = \frac{n}{\ln n}$. Using L'Hôpital's rule to compute the limit,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n}{\ln n} = \lim_{n\to\infty}\frac{1}{1/n}$$
$$= \lim_{n\to\infty} n = \infty \ne 0$$

Thus, condition (2) is not satisfied, and the alternating series $\sum_{n=2}^{\infty}\frac{n(-1)^n}{\ln n}$ is divergent.

(b) Note that $\sum_{n=1}^{\infty}\cos(n\pi) = \sum_{n=1}^{\infty}(-1)^n$.

Then, the series can be written as $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{3/4}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$.

Checking both conditions of the alternating series test.

Let $b_n = \frac{1}{n^{3/4}}$. Clearly, $b_n > 0$.

Condition (1):

Let $f(x) = \frac{1}{x^{3/4}}$. Then, $f'(x) = -\frac{3}{4x^{7/4}} < 0$ for all x values in $[1,\infty)$. This means that b_n is decreasing for all n in $[1,\infty)$.

Condition (2):

Remember the fact:

$\lim_{n\to\infty} \frac{1}{n^p} = 0$, where $p > 0$ is a real number.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^{3/4}} = 0$$

Both conditions of the alternating series test are satisfied. Thus,

$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{3/4}}$ is convergent.