Partial Derivatives

Partial Derivatives

Suppose $z = f(x, y)$. The partial derivatives of f with respect to x is denoted by

$$\frac{\partial z}{\partial x} \quad \text{or} \quad f_x(x, y) \text{ or } \frac{\partial f}{\partial x}$$

and is the function obtained by differentiating f with respect to x, treating y as a constant. The partial derivative of f with respect to y is denoted by

$$\frac{\partial z}{\partial y} \quad \text{or} \quad f_y(x, y) \text{ or } \frac{\partial f}{\partial y}$$

and is the function obtained by differentiating f with respect to y, treating x as a constant.

Computation of Partial Derivatives

No new rules are needed for the computation of partial derivatives.

Example. Find the partial derivatives f_x and f_y, if $f(x, y) = x^2 + 2xy^2 + \frac{2y}{3x}$

Solution.
$$f_x(x, y) = 2x + 2(1)y^2 + \frac{2}{3}y(-x^{-2}) = 2x + 2y^2 - \frac{2y}{3x^2}$$
$$f_y(x, y) = 0 + 2x(2y) + \frac{2}{3}(1)(x^{-1}) = 4xy + \frac{2}{3x}$$

Example. Find the partial derivatives f_x and f_y, if $f(x, y) = xe^{-2xy}$.

Solution.
$$f_x(x, y) = x(-2ye^{-2xy}) + e^{-2xy} = e^{-2xy}(1 - 2xy)$$

$$f_y(x,y) = x(-2xe^{-2xy}) = -2x^2e^{-2xy}$$

Second-Order Partial Differentiation

If $z = f(x,y)$, the partial derivative of f_x with respect to x is

$$f_{xx} = (f_x)_x \quad \text{or} \quad \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{x}\left(\frac{\partial z}{\partial x}\right) \quad \text{or} \quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{x}\left(\frac{\partial f}{\partial x}\right).$$

The partial derivative of f_x with respect to y is

$$f_{xy} = (f_x)_y \quad \text{or} \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{y}\left(\frac{\partial z}{\partial x}\right) \quad \text{or} \quad \frac{\partial^2 f}{\partial yx} = \frac{\partial}{y}\left(\frac{\partial f}{\partial x}\right).$$

The partial derivative of f_y with respect to x is

$$f_{yx} = (f_y)_x \quad \text{or} \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{x}\left(\frac{\partial z}{\partial y}\right) \quad \text{or} \quad \frac{\partial^2 f}{\partial xy} = \frac{\partial}{x}\left(\frac{\partial f}{\partial y}\right).$$

The partial derivative of f_y with respect to y is

$$f_{yy} = (f_y)_y \quad \text{or} \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{y}\left(\frac{\partial z}{\partial y}\right) \quad \text{or} \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{y}\left(\frac{\partial f}{\partial y}\right).$$

The two partial derivatives f_{xy} and f_{yx} are sometimes called mixed second-order partial derivatives of f. If both mixed second-order partial derivatives are continuous (on some disk), then they are equal, i.e., mixed partial derivatives are equal $f_{xy} = f_{yx}$.

Example. Compute the four second-order partial derivatives of the function $f(x,y) = xy^3 + 5xy^2 + 2x + 1.$

Solution. $f_x = y^3 + 5y^2 + 2$. Then, it follows that

$f_{xx} = 0$ and $f_{xy} = 3y^2 + 10y$

$f_y = 3xy^2 + 10xy$. Then, it follows that

$f_{yy} = 6xy + 10x$ and $f_{yx} = 3y^2 + 10y$

Example. Compute the four second-order partial derivatives of the function $f(x, y) = x^2 y e^x$.

Solution. $f_x = y(2xe^x + x^2 e^x)$. Then, it follows that

$$f_{xx} = y[2(e^x + xe^x) + 2xe^x + x^2 e^x]$$
$$= y(4xe^x + 2e^x + x^2 e^x)$$

and $f_{xy} = 2xe^x + x^2 e^x$.

$f_y = x^2 e^x$. Then, it follows that

$f_{yy} = 0$ and $f_{yx} = 2xe^x + x^2 e^x$.