The Robert Gillespie ACADEMIC SKILLS CENTRE

Improper Integrals

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Type I. Infinite Intervals

(a) If f is continuous on $[a,\infty)$, then $\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$.

(b) If f is continuous on $(-\infty,a]$, then $\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx$.

(c) If f is continuous on $(-\infty,\infty)$, then break up the integral at a convenient value of a, i.e., write

$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$, and then use (a) and (b).

Type II. Discontinuous Functions

(a) If f is continuous on $[a,b)$ and discontinuous at b, then

$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$.

(b) If f is continuous on $(a,b]$ and discontinuous at a, then

$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$.

(c) If f is continuous on $[a,b]$, except at a number c in (a,b), then break up the integral at c, i.e., write

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, and then use (a) and (b).

The improper integral is said to:

- **CONVERGE** if the limit in (a) and (b) exists, or if both limits in (c) exist
- **DIVERGE** if the limit in (a) or (b) does not exist, or if at least one of the limits in (c) does not exist

Remember: The improper integral $\int_1^\infty \frac{1}{x^p}\,dx$, where p is a real number converges when $p > 1$, and diverges when $p \le 1$.

Example. Determine whether the following integrals converge or diverge.

(a) $\int_2^\infty e^{-3x}dx$ (b) $\int_{-\infty}^\infty xe^{x^2}\,dx$

(c) $\int_0^1 \frac{1}{x}dx$

Solution. (a)

$$\int_2^\infty e^{-3x}\,dx = \lim_{t\to\infty}\int_2^\infty e^{-3x}\,dx = \lim_{t\to\infty}\left(-\frac{e^{3x}}{3}\right)\Bigg|_2^t$$

$$= \lim_{t\to\infty}\left[\left(-\frac{e^{-3t}}{3}\right) - \left(-\frac{e^{-6}}{3}\right)\right] = \frac{1}{3e^6}$$

Thus, $\int_2^\infty e^{-3x}\,dx$ converges.

(b)

$$\int_{-\infty}^\infty xe^{x^2}\,dx = \int_{-\infty}^0 xe^{x^2}\,dx + \int_0^\infty xe^{x^2}\,dx$$

$$= \lim_{t\to-\infty}\int_t^0 xe^{x^2}\,dx + \lim_{s\to\infty}\int_0^s xe^{x^2}\,dx$$

$$= \lim_{t\to-\infty}\frac{e^{x^2}}{2}\Bigg|_t^0 + \lim_{s\to\infty}\frac{e^{x^2}}{2}\Bigg|_0^s$$

$$= \lim_{t\to-\infty}\left(\frac{1}{2} - \frac{e^{s^2}}{2}\right) + \lim_{s\to\infty}\left(\frac{e^{s^2}}{2} - \frac{1}{2}\right)$$

The first limit is equal to $-\infty$, and thus $\int_{-\infty}^{\infty} xe^{x^2}\,dx$ diverges. (Note that the second limit is $+\infty$ which also implies that the given improper integral diverges.)

In the calculation, note that $\lim_{x\to+\infty} e^x = +\infty$.

(c)

$$\int_0^1 \frac{1}{x}dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}dx = \lim_{t\to 0}\ln x\Big|_t^1$$
$$= \lim_{t\to 0+}(\ln(1) - \ln(t)) = +\infty$$

Thus, $\int_0^1 x\ln x\,dx$ diverges.

Note that $\lim_{x\to 0^+}\ln x = -\infty$.