The Robert Gillespie ACADEMIC SKILLS CENTRE

Separable Differential Equation (DE)

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$$N(x)+M(y)\frac{dy}{dx}=0$$, where $N(x)$ is a function of x only and $M(y)$ is a function of y only.

To solve: Separate the equation so that all y's are on one side and all x's are on the other side. Then, integrate.

Example. Find the general solution of $(x^2+4)\frac{dy}{dx}=xy$.

Solution. Note that $y=0$ is a solution. To find the other solutions, assume that $y\neq 0$ and separate the variable as follows:

$$(x^2+4)\frac{dy}{dx}=xy$$

$$\frac{1}{y}dy=\frac{x}{x^2+4}dx$$

Now, integrate, and try to find an explicit equation for y (if possible).

$$\int \frac{1}{y}dy = \int \frac{x}{x^2+4}dx$$

$$\ln|y| = \frac{1}{2}\ln\left|x^2+4\right| + C$$

$$e^{\ln|y|} = e^{\frac{1}{2}\ln\left|x^2+4\right|+C}$$

$$|y| = e^{\ln\left|x^2+4\right|^{\frac{1}{2}}}e^{C}$$

$$y = \pm A\sqrt{x^2+4}\,, \quad \text{where } A = e^{C}$$

First Order Linear Differential Equation (DE)

$\frac{dy}{dx} + P(x)y = Q(x)$, where $P(x)$ and $Q(x)$ are continuous functions.

To solve: The DE must be in the above form with only the coefficient of 1 in front of $\frac{dy}{dx}$. Compute the integrating factor, $I(x) = e^{\int P(x)dx}$. Multiply the integrating factor across the DE and integrate.

Example. Solve the differential equation $\frac{x}{2}y' + y = 6x^2$, where $x > 0$.

Solution. DE rewritten:

$$y' + \frac{2}{x}y = 12x$$

Integrating factor:

$$I(x) = e^{\int P(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x}$$
$$= e^{\ln(x^2)} = x^2$$

Multiply the DE by the integrating factor, and simplify.

$$x^2\left(y'+\frac{2}{x}y\right)=x^2(12x)$$

$$x^2y'+2xy=12x^3$$

$$(yx^2)'=12x^3$$

The main idea is to recognize that the left-hand side of the DE is the product rule applied to the product of the unknown function y and the integrating factor $I(x)$, that is, $(yx^2)'=y'x^2+2xy$.

Now, integrate the DE, and find an explicit equation for y.

$$\int (yx^2)'dx=\int 12x^3\,dx$$

$$yx^2=3x^4+C$$

$$y=\frac{1}{x^2}(3x^4+C)=3x^2+\frac{C}{x^2}$$