A Magic Trick Using the SET Deck and De Bruijn Sequence

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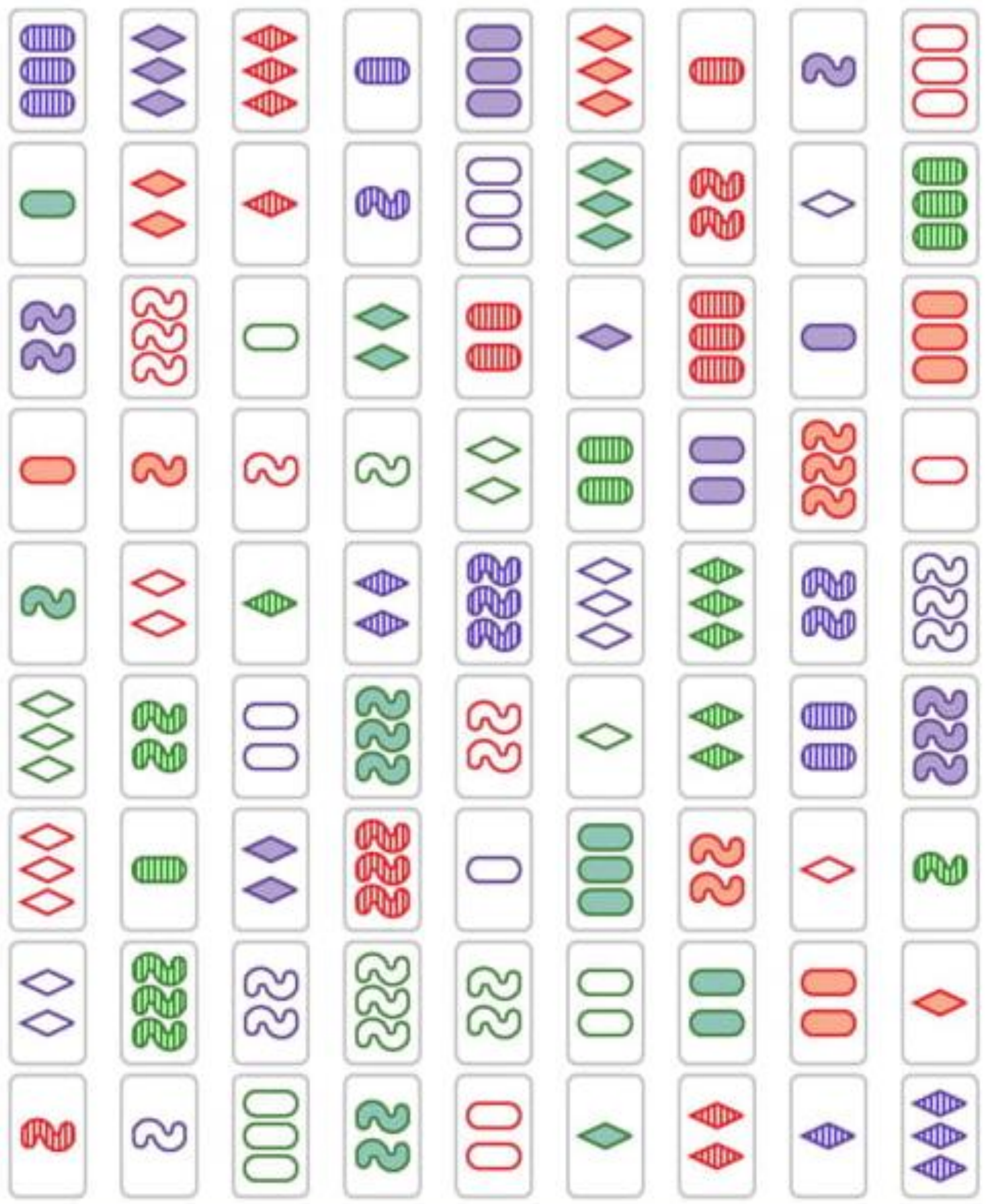
Introduction to the game of SET

- The deck consists of 81 unique cards.
- They vary in four features across three possibilities for each:
  - Number: three, two, one
  - Color: purple, red, green
  - Filling: striped, solid, unfilled
  - Shape: Diamond, Oval, Squiggle

Provide context to alphabet A

Number	Color	Filling	Shape
Three → 0	Purple → 0	Striped → 0	Diamond → 0
One → 1	Red → 1	Solid → 1	Oval → 1
Two → 2	Green → 2	Unfilled → 2	Squiggle → 2

Deck Arrangement for "In TetraCycles"



De Bruijn Sequences and Its Special Properties

Definition

A de Bruijn sequence is a cyclic sequence of a given alphabet A with size k, for which every possible subsequence of length n in A appears as a sequence of consecutive characters exactly once. We denote a de Bruijn sequence as $B(k, n)$.

Example

For example, one element of $B(3, 3)$, using alphabet $\{0, 1, 2\}$, is 021112120002011001221010222.
You can check that each of the 27 strings of length 3 from the alphabet $\{0, 1, 2\}$ appear once and only once as a consecutive subsequence (note that 220 and 202 start at the end and wrap around).

De Bruijn Graphs

One way to build de Bruijn sequences (and thus prove their existence) comes from graph theory.

Definition

Let A be an alphabet with k elements. The de Bruijn graph $G(k, n)$ is a directed graph with k^{n-1} vertices labeled with all possible strings of length n–1 from A. For each vertex v, labeled $a_1a_2 \ldots a_{n-1}$, and each element d of A, there is a directed edge (labeled d) from v to the vertex labeled $a_2a_3 \ldots a_{n-1}d$. In particular, we have an edge labeled d from vertex v to vertex w if the label on w can be obtained by appending d to the label on vertex v with the first character removed.

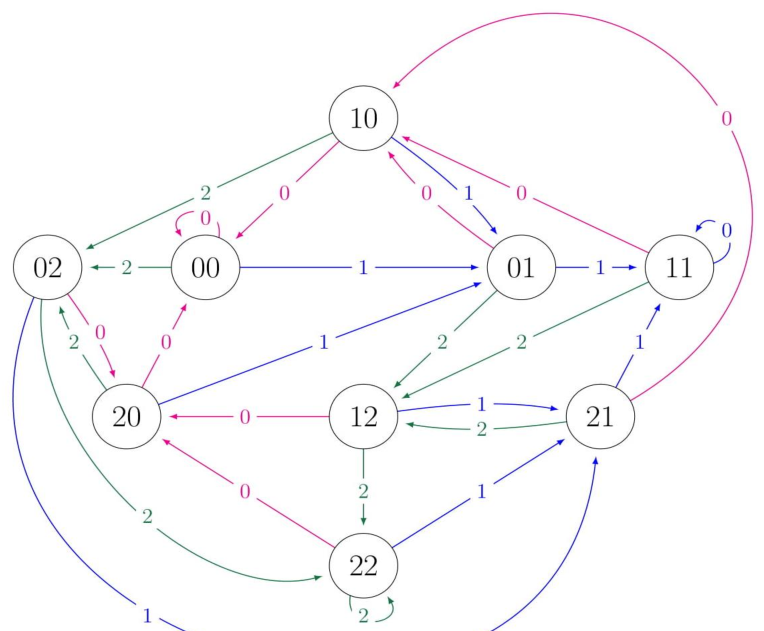
Definition

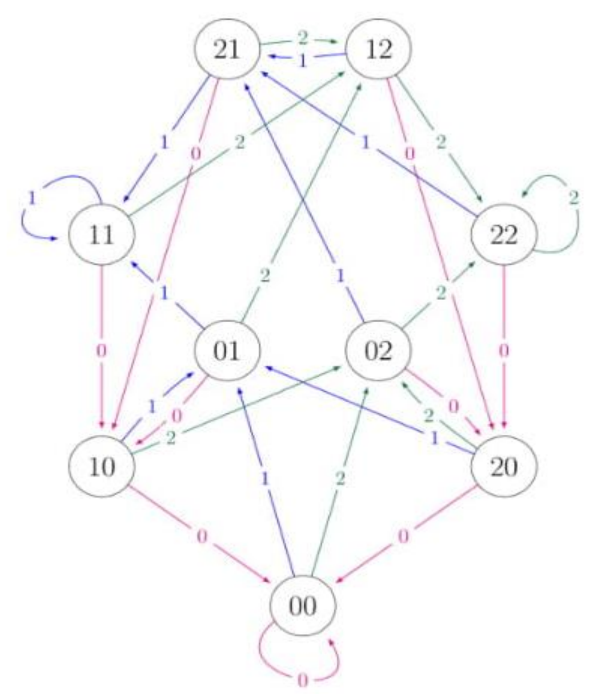
An Eulerian circuit or Eulerian cycle is a trail in a finite graph that visits every edge exactly once and starts and ends on the same vertex.

Traversing through an Eulerian circuit will give us a de Bruijn sequence.

Example

One Eulerian circuit in $G(3, 3)$ is given by the vertex sequence $22 \to 20 \to 02 \to 21 \to 11 \to 11 \to 12 \to 21 \to 12 \to 20 \to 00 \to 00 \to 02 \to 20 \to 01 \to 11 \to 10 \to 00 \to 01 \to 12 \to 22 \to 21 \to 10 \to 01 \to 10 \to 02 \to 22$. When we read off the labels of the associated edges in order, we obtain the de Bruijn sequence from B(3,3) previously mentioned: 021112120002011001221010222.





Recursive De Bruijn Sequences

We will start with 0001, then the fifth digit is $0 + 0 \pmod 3 = 0$, the sixth digit is $0 + 0 \pmod 3 = 0$, and the seventh digit is $0 + 1 \pmod 3 = 1$. By continuing this recursive process, we end up with a complete de Bruijn sequence of length 81:

The Recursive de Bruijn Sequence for "In Tetracycles"

$B =$ 000100110121100210201221010111122201121200020022021220012010211202022221110221210

"In TetraCycles" Walkthrough

1.

- Arrange the deck in a specific order based on our de Bruijn sequence.
- 0001 would be the first card, and 0010 would be the second and so on.

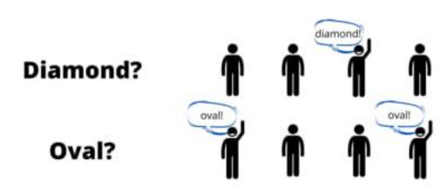
The Recursive de Bruijn Sequence for "In Tetracycles"

$B =$ 000100110121100210201221010111122201121200020022021220012010211202022221110221210

2.

- The magician cuts the deck repeatedly (since it is cyclic, the property of de Bruijn sequence remains).
- The magician hands out four consecutive cards to four volunteers.
- The magician asks "Which feature would you four like to describe?" Let us assume the volunteers choose shape.
- The magician then asks "Would all of you holding the diamond please raise your hand?" Let us assume the third person raises their hand.
- The magician asks "Now, would everyone with an oval please raise your hand?" Let us assume the first and fourth person raise their hand.
- The magician correctly name all four cards in order, including their numbers, colors, shadings, and shapes.

3.



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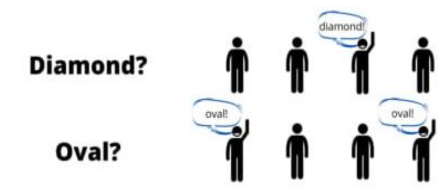
- We deduct that the one volunteer that did not raise the hand must have a card with squiggle shape.
- The order "Oval, Squiggle, Diamond, Oval" corresponds to 1201.
- Since shape is the last feature in our table, to know the complete information of all four cards, we need to go backwards and deduct $B_iB_{i+i}B_{i+2}$1201.

4.



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- By our recursive definition, we have:
  - $B_{i+6} - B_{i+3} = 1 - 1 = 0$, so $B_{i+2} = 0$
  - $B_{i+5} - B_{i+2} = 0 - 0 = 0$, so $B_{i+1} = 0$
  - $B_{i+4} - B_{i+1} = 2 - 0 = 2$, so $B_i = 2$
- Thus, we have an entire length seven subsequence of the form 2001201, from which we can read off the algebraic repres[illegible] each card in order.

5.

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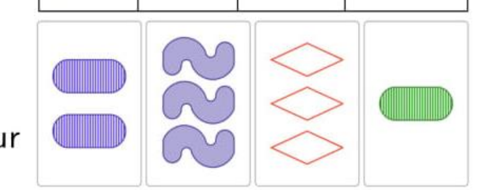

- The sequence 2001201 provides us complete information of the four cards that the volunteers are holding:
  - 1st card is 2001
  - 2nd card is 0012
  - 3rd card is 0120
  - 4th card is 1201
- By applying the table and match each number with its corresponding properties, we obtain the four cards on the left, thus complete the trick.

Acknowledgement

***Zhengyu Li** is an undergraduate student at the University of Toronto Mississauga. He enjoys exploring new mathematical card tricks.*

***Parker Glynn-Adey** is an assistant professor of mathematics, Teaching Stream, at the University of Toronto Mississauga. He works in mathematics education, combinatorics, and geometry.*

In TetraCycles is a variation on the In Cycles trick developed by Persi Diaconis and Ron Graham and elaborated in their book Magical Mathematics: The Mathematical Ideas That Animate Great Magic Tricks (2015). Reading Chapter 3 on "Universal Cycles" got us started on this exploration. We dedicate this work to the mathematical legacy of Dr. Ronald Graham.
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