Integration Bee 2023

Instructions

1. Solve as many integrals as you can within the time limit!
2. Each problem is worth **one point**.
3. You may work on the problems in any order you like.
4. Don't forget $+C$!
5. **Show your work!** A correct answer with no admissible justification will not count for points. The Integration Judges have the final say in how much work counts as "admissible".
6. Allowed materials: writing utensils, and blank paper for rough work. (No calculators, computers, or notes!)
7. Ties will be broken using additional problems.
8. **Time limit: 45 minutes**.

Integrals

1. $\displaystyle\int \frac{e^{1/x}}{(xe^{1/x}+x)^2}\,dx$
2. $\displaystyle\int_0^1 \sqrt{1-\sqrt{x}}\,dx$
3. $\displaystyle\int \cos(\ln(x))\,dx$
4. $\displaystyle\int_1^{10} \lfloor \ln\lfloor x\rfloor\rfloor\,dx$
5. $\displaystyle\int \left(\frac{1}{x+\frac{1}{x}}\right)^2 dx$
6. $\displaystyle\int \sqrt[3]{\tan(x)}\,dx$
7. $\displaystyle\int_0^4 \frac{1}{|x-2|+|x-3|}\,dx$
8. $\displaystyle\int \sqrt{x+\sqrt{x+\sqrt{x+\cdots}}}\,dx$
9. $\displaystyle\int_0^1 \ln(1+\sqrt{x})\,dx$

Tie Breakers

1. Let $f:\mathbf{R}\to\mathbf{R}$ be a continuous function satisfying $f(2x)=3f(x)$ for all $x\in\mathbf{R}$, and $\displaystyle\int_0^1 f(x)\,dx=1$.

   Calculate $\displaystyle\int_1^2 f(x)\,dx$.
2. Evaluate the integral: $\displaystyle\int_0^{\pi} |\sin(20x)|\,dx$.
3. Let $f(x)=x^3+x+1$. Evaluate the integral: $\displaystyle\int_1^3 f^{-1}(y)\,dy$.

Solutions

1. *Answer:* $\dfrac{1}{e^{1/x}+1}+C.$

   Solution sketch: Substitute $u = 1/x$, then substitute $v = e^u + 1$.

2. *Answer:* 8/15.

   Solution sketch: Substitute $u = 1-\sqrt{x}$ and use $dx = 2(u-1)\,du$ to simplify the integral into $2\int_1^0 u^{3/2} - u^{1/2}\,du$.

3. *Answer:* $\dfrac{x}{2}(\sin\ln(x) + \cos\ln(x)) + C.$

   Solution sketch: Substitute $u \ln(x)$ and rearrange to get $dx = e^u\,du$. The integral becomes $\int e^u \cos(u)\,du$, which is a classic integral — solve it by using integration by parts twice.

4. *Answer:* 9.

   Solution sketch: The flooring operation turns this into a "staircase" function which only takes integer values. The key points of interest are when $x = 1$, $x = e \approx 2.71$, and $x = e^2 \approx 7.38$. So we split the x-axis up into three cases based on this.

   - If $1 \le x < 3$, then $1 \le \lfloor x \rfloor \le 2$, so $0 \le \ln\lfloor x \rfloor \le \ln(2)$. Note $\ln(2)$ is smaller than $\ln(e) = 1$. Thus $\ln\lfloor x \rfloor$ is in $[0, 1)$, which means $\lfloor \ln\lfloor x \rfloor \rfloor = 0$.
   - If $3 \le x < 8$, then $3 \le \lfloor x \rfloor \le 7$, so $\ln(3) \le \ln\lfloor x \rfloor \le \ln(7)$. Note that $\ln(7)$ is strictly smaller than $\ln(e^2) = 2$, and $\ln(3)$ is strictly larger than $\ln(e) = 1$. Thus,

     $$1 < \ln\lfloor x \rfloor < 2.$$

     We conclude that $\lfloor \ln\lfloor x \rfloor \rfloor = 1$ in this case.
   - If $8 \le x \le 10$, then similar analysis shows $\lfloor \ln\lfloor x \rfloor \rfloor = 2$.

   Thus the integral can be split up over these three domains, to get:

   $$\int_1^{10} \lfloor \ln\lfloor x \rfloor \rfloor\,dx = \int_1^3 0\,dx + \int_3^8 1\,dx + \int_8^{10} 2\,dx = 9.$$

5. *Answer:* $\dfrac{1}{2}\arctan(x) - \dfrac{x}{2(x^2+1)} + C.$

   Solution sketch: The integrand becomes

   $$\left(\frac{1}{x+1/x}\right)^2 = \frac{x^2}{(x^2+1)^2} = \frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}.$$

   The first term is $\arctan(x)$. The second term requires a trig substitution with $x = \tan(\theta)$, which simplifies into $\int \cos^2(\theta)\,d\theta$. This one can be solved with the half-angle identity.

6. *Answer:* $\dfrac{1}{4}\ln\left(\tan^{4/3}(x) - \tan^{2/3}(x) + 1\right) + \dfrac{\sqrt{3}}{2}\arctan\left(\dfrac{2\tan^{2/3}(x) - 1}{\sqrt{3}}\right) - \dfrac{1}{4}\ln\left(\tan^{2/3}(x) + 1\right) + C.$

   Solution sketch: Substitute $u = \tan(x)$. Then $du = \sec^2(x)\,dx = (\tan^2(x)+1)\,dx = (u+1)\,dx$, so our integral transforms as follows:

   $$\int \tan^{1/3}(x)\,dx = \int \frac{u^{1/3}}{u+1}\,du.$$

   This can be turned into a rational function using the substitution $v = u^{2/3}$, after which you can use partial fractions.

7. *Answer:* $1 + \dfrac{1}{2}\ln(15) \approx 2.3540.$

   Solution sketch: Split into three cases: $0 \le x \le 2$, $2 \le x \le 3$, and $3 \le x \le 4$. The integrand simplifies greatly in each case.

8. *Answer:* $\frac{1}{2}x + \frac{1}{12}(4x+1)^{3/2} + C$.

   Solution sketch: Let $f(x) = \sqrt{x + \sqrt{x + \cdots}}$ be the integrand in question. Notice that $f(x)$ satisfies the recurrence $\sqrt{x + f(x)} = f(x)$. Solving for $f(x)$ here gives

$$f(x) = \frac{1 + \sqrt{1+4x}}{2}$$

   which is easy enough to integrate.

9. *Answer:* 1/2.

   Solution sketch: Use integration by parts with $u = \ln(1 + \sqrt{x})$ and $dv = dx$. The new integral you end up with is:

$$\int \frac{\sqrt{x}}{1+\sqrt{x}}\, dx.$$

   This can be simplified into a rational function after substituting $u = 1 + \sqrt{x}$: you get

$$\int \frac{(u-1)^2}{u}\, du$$

   which is easy enough to solve, by expanding the numerator.