Physics Club – Math Preparation for Introductory Physics

Ilona Fernandes

Table of Contents

Algebra and Trigonometry

A-1 Algebra and Trigonometry

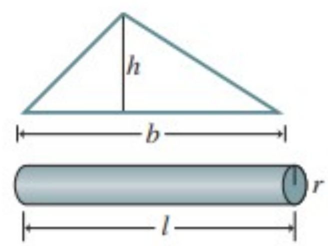
Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Circumference, Area, Volume

Where $\pi \simeq 3.14159\ldots$:

circumference of circle	$2\pi r$
area of circle	πr^2
surface area of sphere	$4\pi r^2$
volume of sphere	$\frac{4}{3}\pi r^3$
area of triangle	$\frac{1}{2}bh$
volume of cylinder	$\pi r^2 l$

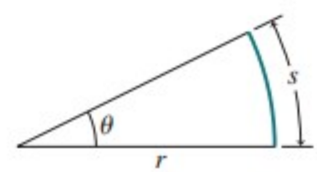


Trigonometry

definition of angle (in radians): $\theta = \dfrac{s}{r}$

2π radians in complete circle

1 radian $\simeq 57.3°$

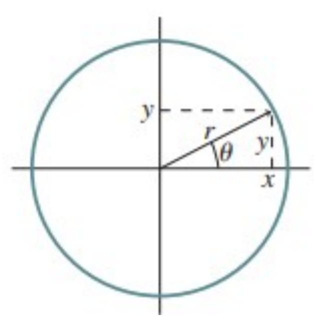


Trigonometric Functions

$\sin\theta = \dfrac{y}{r}$

$\cos\theta = \dfrac{x}{r}$

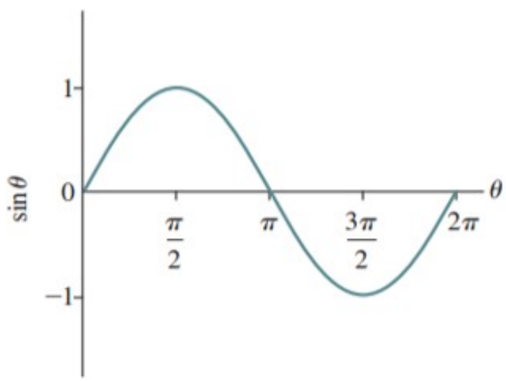
$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{y}{x}$

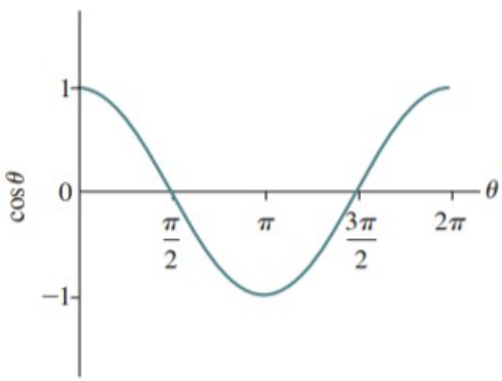


Values at Selected Angles

$\theta \rightarrow$	0	$\frac{\pi}{6}$ (30°)	$\frac{\pi}{4}$ (45°)	$\frac{\pi}{3}$ (60°)	$\frac{\pi}{2}$ (90°)
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞

Graphs of Trigonometric Functions





Trigonometric Identities

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\sin\left(\theta \pm \frac{\pi}{2}\right) = \pm\cos\theta$

$\cos\left(\theta \pm \frac{\pi}{2}\right) = \mp\sin\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\sin\alpha \pm \sin\beta = 2\sin\left[\frac{1}{2}(\alpha \pm \beta)\right]\cos\left[\frac{1}{2}(\alpha \mp \beta)\right]$

$\cos\alpha + \cos\beta = 2\cos\left[\frac{1}{2}(\alpha + \beta)\right]\cos\left[\frac{1}{2}(\alpha - \beta)\right]$

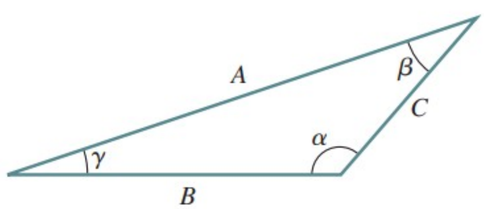
$\cos\alpha - \cos\beta = -2\sin\left[\frac{1}{2}(\alpha + \beta)\right]\sin\left[\frac{1}{2}(\alpha - \beta)\right]$

Law of Cosines and Sines

Where A, B, C are the sides of an arbitrary triangle and α, β, γ the angles opposite those sides:

Law of cosines

$C^2 = A^2 + B^2 - 2AB\cos\gamma$



Law of sines

$$\frac{\sin\alpha}{A} = \frac{\sin\beta}{B} = \frac{\sin\gamma}{C}$$

Exponentials and Logarithms

$$e^{\ln x} = x, \quad \ln e^x = x \quad e = 2.71828\ldots$$

$$a^x = e^{x \ln a} \qquad \ln(xy) = \ln x + \ln y$$

$$a^x a^y = a^{x+y} \qquad \ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

$$(a^x)^y = a^{xy} \qquad \ln\left(\frac{1}{x}\right) = -\ln x$$

$$\log x \equiv \log_{10} x = \ln(10)\ln x \simeq 2.3 \ln x$$

Approximations

For $|x| \ll 1$, the following expressions provide good approximations to common functions:

$e^x \simeq 1 + x$

$\sin x \simeq x$

$\cos x \simeq 1 - \frac{1}{2}x^2$

$\ln(1 + x) \simeq x$

$(1 + x)^p \simeq 1 + px \quad$ (binomial approximation)

Expressions that don't have the forms shown may often be put in the appropriate form. For example:

$$\frac{1}{\sqrt{a^2 + y^2}} = \frac{1}{a\sqrt{1 + \dfrac{y^2}{a^2}}} = \frac{1}{a}\left(1 + \frac{y^2}{a^2}\right)^{-1/2} \simeq \frac{1}{a}\left(1 - \frac{y^2}{2a}\right) \quad \text{for} \quad y^2/a^2 \ll 1, \text{ or } y^2 \ll a^2$$

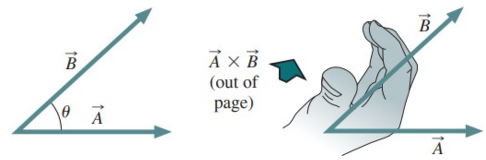
Vectors

Vector Algebra

Vector Products

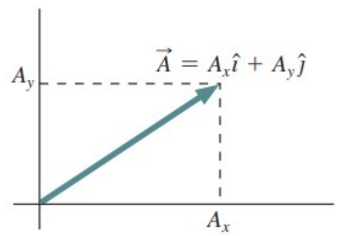
$\vec{A}\cdot\vec{B} = AB\cos\theta$

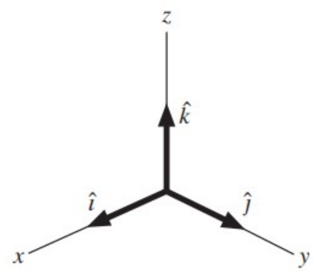
$|\vec{A}\times\vec{B}| = AB\sin\theta$, with direction of $\vec{A}\times\vec{B}$ given by the right-hand rule:



Unit Vector Notation

An arbitrary vector $\vec{A}$ may be written in terms of its components A_x, A_y, A_z and the unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ that have magnitude 1 and lie along the x-, y-, z-axes:





In unit vector notation, vector products become

$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$

$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{\imath} + (A_zB_x - A_xB_z)\hat{\jmath} + (A_xB_y - A_yB_x)\hat{k}$

Vector Identities

$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$

$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$

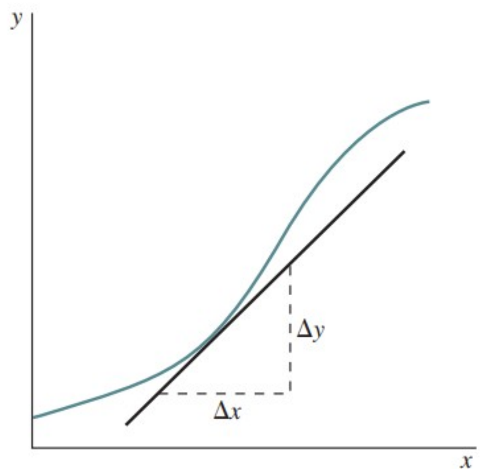
Derivatives

Definition of the Derivative

If y is a function of x, then the **derivative of y with respect to x** is the ratio of the change Δy in y to the corresponding change Δx in x, in the limit of arbitrarily small Δx:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Algebraically, the derivative is the rate of change of y with respect to x; geometrically, it is the slope of the y versus x graph—that is, of the tangent line to the graph at a given point:



Derivatives of Common Functions

$$\frac{da}{dx} = 0 \quad (a \text{ is a constant})$$

$$\frac{dx^n}{dx} = nx^{n-1} \quad (n \text{ need not be an integer})$$

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}$$

$$\frac{de^x}{dx} = e^x$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Derivatives of Sums, Products, and Functions of Functions

1. Derivative of a constant times a function

$$\frac{d}{dx}[af(x)] = a\frac{df}{dx} \qquad (a \text{ is a constant})$$

2. Derivative of a sum

$$\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx} + \frac{dg}{dx}$$

3. Derivative of a product

$$\frac{d}{dx}[f(x)g(x)] = g\frac{df}{dx} + f\frac{dg}{dx}$$

Examples

$$\frac{d}{dx}(x^2\cos x) = \cos x\frac{dx^2}{dx} + x^2\frac{d}{dx}\cos x = 2x\cos x - x^2\sin x$$

$$\frac{d}{dx}(x\ln x) = \ln x\frac{dx}{dx} + x\frac{d}{dx}\ln x = (\ln x)(1) + x\left(\frac{1}{x}\right) = \ln x + 1$$

To find the derivative of $f(x) = 3x^2 + 2x$,

$$\begin{aligned}\frac{d}{dx}[3x^2 + 2x] &= \frac{d}{dx}[3x^2] + \frac{d}{dx}[2x] \\ &= 3\frac{d}{dx}[x^2] + 2\frac{d}{dx}[x] \\ &= 3[2x] + 2[1] \\ &= 6x + 2\end{aligned}$$

Example of using the Product Rule for derivatives:

Example:

Find f'(x) if f(x) = $(6x^3)(7x^4)$

Solution:

Using the Product Rule, we get

$$\begin{aligned} f'(x) &= \left(6x^3\right)\frac{d}{dx}\left(7x^4\right) + \left(7x^4\right)\frac{d}{dx}\left(6x^3\right) \\ &= \left(6x^3\right)\left(28x^3\right) + \left(7x^4\right)\left(18x^2\right) \\ &= 168x^6 + 126x^6 = 294x^6\end{aligned}$$

4. Derivative of a quotient

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{1}{g^2}\left(g\frac{df}{dx} - f\frac{dg}{dx}\right)$$

Example

$$\frac{d}{dx}\left(\frac{\sin x}{x^2}\right) = \frac{1}{x^4}\left(x^2\frac{d}{dx}\sin x - \sin x\frac{dx^2}{dx}\right) = \frac{\cos x}{x^2} - \frac{2\sin x}{x^3}$$

5. Chain rule for derivatives

If f is a function of u and u is a function of x, then

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$$

Example of using the Quotient Rule for derivatives:

$$W(z) = \frac{3z+9}{2-z}$$

There isn't a lot to do here other than to use the quotient rule. Here is the work for this function.

$$W'(z) = \frac{3(2-z) - (3z+9)(-1)}{(2-z)^2} = \frac{15}{(2-z)^2}$$

Example of using the chain rule for derivatives:

(a) $f(x) = \sin(3x^2 + x)$

It looks like the outside function is the sine and the inside function is $3x^2+x$. The derivative is then.

$$f'(x) = \underbrace{\cos}_{\text{derivative of outside function}} \underbrace{(3x^2+x)}_{\text{leave inside function alone}} \underbrace{(6x+1)}_{\text{times derivative of inside function}}$$

Or with a little rewriting,

$$f'(x) = (6x+1)\cos(3x^2+x)$$

a. Evaluate $\frac{d}{dx}\sin(x^2)$.

b. $\frac{d}{dt}\sin\omega t$

c. Evaluate $\frac{d}{dx}\sin^2 5x$.

Second Derivatives

The second derivative of y with respect to x is defined as the derivative of the derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Example

If $y = ax^3$, then $dy/dx = 3ax^2$, so

$$\frac{d^2y}{dx^2} = \frac{d}{dx}3ax^2 = 6ax$$

Partial Derivatives

Partial Derivatives

When a function depends on more than one variable, then the partial derivatives of that function are the derivatives with respect to each variable, taken with all other variables held constant. If f is a function of x and y, then the partial derivatives are written

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}$$

Example

If $f(x, y) = x^3 \sin y$, then

$$\frac{\partial f}{\partial x} = 3x^2 \sin y \quad \text{and} \quad \frac{\partial f}{\partial y} = x^3 \cos y$$

Example 1

Let $f(x,y) = y^3x^2$. Calculate $\frac{\partial f}{\partial x}(x,y)$.

Solution:

$$\frac{\partial f}{\partial x}(x,y) = 2y^3x.$$

Example 2

For the same f, calculate $\frac{\partial f}{\partial y}(x,y)$.

Solution:

$$\frac{\partial f}{\partial y}(x,y) = 3x^2y^2.$$

Practice Problems for Derivatives:

1. $f(x) = 4x^5 - 5x^4$
2. $f(x) = e^x \sin x$
3. $f(x) = (x^4 + 3x)^{-1}$
4. $f(x) = 3x^2(x^3 + 1)^7$
5. $f(x) = \cos^4 x - 2x^2$
6. $f(x) = \dfrac{x}{1 + x^2}$
7. $f(x) = \dfrac{x^2 - 1}{x}$
8. $f(x) = (3x^2)(x^{\frac{1}{2}})$
9. $f(x) = \ln(xe^{7x})$
10. $f(x) = \dfrac{2x^4 + 3x^2 - 1}{x^2}$
11. $f(x) = (x^3)\sqrt[5]{2 - x}$
12. $f(x) = 2x - \dfrac{4}{\sqrt{x}}$

For problems 1 - 6 differentiate the given function.

1. $f(x) = 2\mathbf{e}^x - 8^x$
2. $g(t) = 4\log_3(t) - \ln(t)$
3. $R(w) = 3^w \log(w)$
4. $y = z^5 - \mathbf{e}^z \ln(z)$
5. $h(y) = \dfrac{y}{1 - \mathbf{e}^y}$
6. $f(t) = \dfrac{1 + 5t}{\ln(t)}$

For the following problems, find the first-order partial derivatives.

1. $f(x, y, z) = 4x^3y^2 - \mathbf{e}^z y^4 + \frac{z^3}{x^2} + 4y - x^{16}$

2. $w = \cos(x^2 + 2y) - \mathbf{e}^{4x - z^4 y} + y^3$

3. $f(u, v) = u^2 \sin(u + v^3) - \sec(4u)\tan^{-1}(2v)$

Integrals

Indefinite Integrals

Integration is the inverse of differentiation. The **indefinite integral**, $\int f(x)\,dx$, is defined as a function whose derivative is $f(x)$:

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$$

If $A(x)$ is an indefinite integral of $f(x)$, then because the derivative of a constant is zero, the function $A(x) + C$ is also an indefinite integral of $f(x)$, where C is any constant. Inverting the derivatives of common functions listed in the preceding section gives the integrals that follow (a more extensive table appears at the end of this appendix).

$$\int a\,dx = ax + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \qquad \int e^x\,dx = e^x + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int x^{-1}\,dx = \ln x + C$$

Definite Integrals

In physics we're most often interested in the **definite integral**, defined as the sum of a large number of very small quantities, in the limit as the number of quantities grows arbitrarily large and the size of each arbitrarily small:

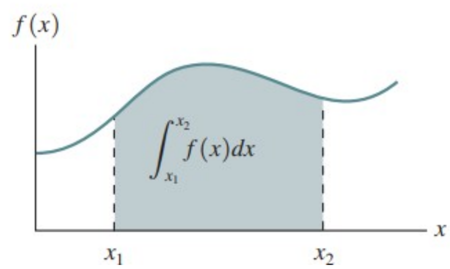
$$\int_{x_1}^{x_2} f(x)\,dx \equiv \lim_{\substack{\Delta x \to 0 \\ N \to \infty}} \sum_{i=1}^{N} f(x_i)\,\Delta x$$

where the terms in the sum are evaluated at values x_i between the limits of integration x_1 and x_2; in the limit $\Delta x \to 0$, the sum is over all values of x in the interval.

The key to evaluating the definite integral is provided by the **fundamental theorem of calculus**. The theorem states that, if $A(x)$ is an *indefinite* integral of $f(x)$, then the *definite integral* is given by

$$\int_{x_1}^{x_2} f(x)\,dx = A(x_2) - A(x_1) \equiv A(x)\Big|_{x_1}^{x_2}$$

Geometrically, the definite integral is the area under the graph of $f(x)$ between the limits x_1 and x_2:



Evaluating Integrals

1. Change of variables

An unfamiliar integral can often be put into familiar form by defining a new variable. For example, it is not obvious how to integrate the expression

$$\int \frac{x\,dx}{\sqrt{a^2 + x^2}}$$

where a is a constant. But let $z = a^2 + x^2$. Then

$$\frac{dz}{dx} = \frac{da^2}{dx} + \frac{dx^2}{dx} = 0 + 2x = 2x$$

so $dz = 2x\,dx$. Then the quantity $x\,dx$ in our unfamiliar integral is just $\frac{1}{2}dz$, while the quantity $\sqrt{a^2 + x^2}$ is just $z^{1/2}$. So the integral becomes

$$\int \frac{1}{2} z^{-1/2}\,dz = \frac{\frac{1}{2} z^{1/2}}{\frac{1}{2}} = \sqrt{z}$$

where we have used the standard form for the integral of a power of the independent variable. Substituting back $z = a^2 + x^2$ gives

$$\int \frac{x\,dx}{\sqrt{a^2 + x^2}} = \sqrt{a^2 + x^2}$$

Example

Suppose we want to find the integral

$$\int (x+4)^5\,\mathrm{d}x$$

We let $u = x + 4$.

$$\int (x+4)^5\,\mathrm{d}x = \int u^5 \mathrm{d}u$$

The resulting integral can be evaluated immediately to give $\dfrac{u^6}{6} + c$.

$$\int (x+4)^5\,\mathrm{d}x = \frac{(x+4)^6}{6} + c$$

Practice Problems for Integrals:

1. $\int 4x^6 - 2x^3 + 7x - 4\,dx$

2. $\int z^7 - 48z^{11} - 5z^{16}\,dz$

3. $\int 10t^{-3} + 12t^{-9} + 4t^3\,dt$

4. $\int w^{-2} + 10w^{-5} - 8\,dw$

5. $\int 12\,dy$

6. $\int \sqrt[3]{w} + 10\,\sqrt[5]{w^3}\,dw$

7. $\int \sqrt{x^7} - 7\,\sqrt[6]{x^5} + 17\,\sqrt[3]{x^{10}}\,dx$

8. $\int \frac{4}{x^2} + 2 - \frac{1}{8x^3}\,dx$

9. $\int \frac{7}{3y^6} + \frac{1}{y^{10}} - \frac{2}{\sqrt[3]{y^4}}\,dy$

10. $\int \left(t^2 - 1\right)\left(4 + 3t\right)\,dt$

Evaluate each of the following integrals, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

2. $\int_1^6 12x^3 - 9x^2 + 2\,dx$

3. $\int_{-2}^1 5z^2 - 7z + 3\,dz$

4. $\int_3^0 15w^4 - 13w^2 + w\,dw$

5. $\int_1^4 \frac{8}{\sqrt{t}} - 12\sqrt{t^3}\,dt$

6. $\int_1^2 \frac{1}{7z} + \frac{\sqrt[3]{z^2}}{4} - \frac{1}{2z^3}\,dz$

Table of Integrals

More extensive tables are available in many mathematical and scientific handbooks; see, for example, *Handbook of Chemistry and Physics* (Chemical Rubber Co.) or Dwight, *Tables of Integrals and Other Mathematical Data* (Macmillan). Some math software, including *Mathematica* and *Maple*, can also evaluate integrals symbolically. Wolfram Research provides *Mathematica*-based integration both at integrals.wolfram.com and through WolframAlpha at www.wolframalpha.com/calculators/integral-calculator.

In the expressions below, a and b are constants. An arbitrary constant of integration may be added to the right-hand side.

$$\int e^{ax}\,dx = \frac{e^{ax}}{a}$$

$$\int \sin ax\,dx = -\frac{\cos ax}{a}$$

$$\int \cos ax\,dx = \frac{\sin ax}{a}$$

$$\int \tan ax\,dx = -\frac{1}{a}\ln(\cos ax)$$

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int x\sin ax\,dx = \frac{1}{a^2}\sin ax - \frac{1}{a}x\cos ax$$

$$\int x\cos ax\,dx = \frac{1}{a^2}\cos ax + \frac{1}{a}x\sin ax$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left(x + \sqrt{x^2 \pm a^2}\right)$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{x\,dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2}$$

$$\int \frac{x\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}$$

$$\int \frac{dx}{(x^2 \pm a^2)^{3/2}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax - 1)$$

$$\int x^2e^{ax}\,dx = \frac{x^2e^{ax}}{a} - \frac{2}{a}\left[\frac{e^{ax}}{a^2}(ax - 1)\right]$$

$$\int \frac{dx}{a + bx} = \frac{1}{b}\ln(a + bx)$$

$$\int \frac{dx}{(a + bx)^2} = -\frac{1}{b(a + bx)}$$

$$\int \ln ax\,dx = x\ln ax - x$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a + x}{a - x}\right|$$

References

Taken from: Wolfson, R. (2020). *Essential university physics* (Fourth edition). Pearson Education.

Practice Problems from:

- https://tutorial.math.lamar.edu/problems/calciii/PartialDerivatives.aspx
- https://www.mathcentre.ac.uk/resources/workbooks/mathcentre/web-integrationbysub-tony.pdf
- https://mathinsight.org/partial_derivative_examples