The Robert Gillespie ACADEMIC SKILLS CENTRE

# Definition of Continuity

## Definition of Continuity

A function $f(x)$ is continuous at a point $x=a$ if and only if the following conditions are satisfied:

1. $f(a)$ exists (i.e., $a$ lies in the domain of $f$)

2. $\lim_{x \to a} f(x)$ exists ($f$ has a limit as $x \to a$, i.e., the limit is a real number)

3. $\lim_{x \to a} f(x) = f(a)$ (the limit equals the function value)

If one or more of these conditions fail, then $f(x)$ is not continuous at a point $x=a$.

Example. Consider the function $f(x)=\begin{cases} \dfrac{x^3+2x^2-8x}{x^2-4} & , \text{ if } x \neq 2 \\ 4 & , \text{ if } x = 2 \end{cases}$.

(a) Is $f(x)$ continuous at $x=-2$?

(b) Does $\lim_{x \to 2} f(x)$ exist?

(c) Is $f(x)$ continuous at $x=2$?

Solution. (a) $f(-2)=\dfrac{(-2)^3+2(-2)^2-8(-2)}{(-2)^2-4}=\dfrac{16}{0}$. Thus, $f(-2)$ is not defined, and hence, $f(x)$ is NOT continuous at $x=-2$.

(b) Note that one-side is not necessary, but for completeness, it is shown.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^3+2x^2-8x}{x^2-4}$$
$$= \lim_{x\to 2^-} \frac{x(x+4)(x-2)}{(x+2)(x-2)} = \lim_{x\to 2^-} \frac{x(x+4)}{x+2}$$
$$= \frac{2(2+4)}{2+2} = 3$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x(x+4)}{x+2} = \frac{2(2+4)}{2+2} = 3$$

Yes, the limit exists, that is, $\lim_{x\to 2} f(x) = 3$.

(c) Note that $f(2) = 4 \neq 3 = \lim_{x\to 2} f(x)$
By the definition of continuity, $f(x)$ is NOT continuous at $x=2$.